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A static condensation reduced basis element method: Complex problems

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ABSTRACT

We extend the static condensation reduced basis element (scRBE) method to treat the class of parametrized complex Helmholtz partial differential equations. The main ingredients are (i) static condensation at the interdomain level, (ii) a conforming eigenfunction "port" representation at the interface level, (iii) the reduced basis (RB) approximation of finite element (FE) bubble functions at the intradomain level, and (iv) rigorous system-level error bounds which reflect RB perturbation of the FE Schur complement. We then incorporate these ingredients in an Offline–Online computational strategy to achieve rapid and accurate prediction of parametric systems formed from instantiations of interoperable parametrized archetype components from a Library. We introduce a number of extensions with respect to the original scRBE framework: first, primal–dual RB methods for general non-symmetric (complex) problems; second, stability constant procedures for weakly coercive problems (at both the interdomain level and intradomain level); third, treatment of non-port linear–functional outputs (as well as functions of outputs); fourth, consideration of particular components and outputs relevant to acoustic applications. We consider several numerical examples in acoustics (in particular focused on mufflers and horns) to demonstrate that the approach can handle models with many parameters and/or topology variations with particular reference to waveguide-like applications.

1. Introduction

Computational simulation of parametrized partial differential equations (PDEs) is an essential tool in many engineering and design contexts. However, classical numerical methods, such as finite element or finite difference methods, are often computationally expensive, and therefore have limited applicability in many applications which require rapid response. Model order reduction methods, such as proper orthogonal decomposition (POD) [1] or the reduced basis (RB) method [2–4], are thus an attractive alternative. However, most model order reduction methods are constructed for a fixed model with predetermined geometry and, in general, are incapable of dealing with many (even $O(10)$) parameters.

The static condensation reduced basis element method (scRBE), introduced in [5], permits the treatment of many parameters—albeit at some additional computational cost. The scRBE is similar in many ways to methods proposed earlier. In particular, the scRBE may be viewed as a parametric extension of component modal synthesis approaches [6–9]; the scRBE may be interpreted as a "strong matching" and hence interoperable variant of the classical reduced basis element approach [10]; and finally the scRBE may be seen as a "heterogeneous" version of the Multiscale RB method [11] which in turn is informed by the Multiscale FE method [12]. Key differentiators of the scRBE with respect to these earlier approaches are, first, interoperability of components for greater flexibility, and second, rigorous and efficient system–level error bounds.

In this paper, we extend the scRBE framework to treat the class of complex, non-symmetric elliptic PDEs. We introduce a number of extensions with respect to the original scRBE framework: first, primal–dual RB methods for general non-symmetric (complex) problems; second, stability constant procedures for weakly coercive problems (at both the interdomain level and intradomain level); third, treatment of non-port linear–functional outputs (as well as functions of outputs); fourth, consideration of particular components and outputs relevant to acoustic applications.

In Section 2 we introduce a component-based framework for our target model problem. The truth approximation, which we wish to accelerate, is introduced in Section 3. In Sections 4 and 5 we develop the static condensation RBE approximation and error bounds, respectively. We then discuss the Offline and Online computational procedures in Section 6, where we provide detailed operation counts and storage requirements for the Online stage in particular. In Section 7 we introduce a complex-valued non-symmetric Helmholtz equation for modeling acoustic devices,

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and in Section 8 we present a series of numerical results for the static condensation RBE framework applied to some acoustic device model problems.

We note that we include in this paper much of the foundation material from [5] as the new contributions and extensions are rather densely interspersed and hence not easily segregated. We adopt as much as possible the language and notation of [5] so that the details we do omit may be readily identified in [5].

2. Component-based framework

We develop a component-based framework [5] for modeling acoustic devices (mufflers and horns). This framework is developed in terms of *physical domain* and *reference domain* quantities. We begin with the former.

2.1. Physical domain

We first introduce a set or library of regions, which we denote "archetype component" domains, $\hat{\Omega}^o_m(\hat{\mu}^{\rm geo}_m), 1 \leqslant m \leqslant M$, for geometry parameters $\hat{\mu}^{\rm geo}_m$. The mth archetype component has $\hat{P}^{\rm geo}_m$ geometric parameters, which reside in an associated geometry parameter domain $\hat{\mathcal{D}}^{\rm geo}_m \subset \mathbb{R}^{\hat{P}^{\rm geo}_m}$. For each archetype component domain $\hat{\Omega}^o_m(\hat{\mu}^{\rm geo}_m)$ we further identify elements of the boundary, $\hat{\gamma}^o_{m,j}(\hat{\mu}^{\rm geo}_m), 1 \leqslant j \leqslant n^\gamma_m$ (where $n^\gamma_m \geqslant 1$), which we denote "archetype component port" domains. We require for simplicity that the intersection of any two ports in any given archetype component is empty; this "mutually disjoint port" condition may be relaxed, as we discuss further below. Here the $^\wedge$ indicates archetype and the ^{o} refers to a quantity defined over the physical domain (we will later introduce reference domains).

We associate to each of these archetype component domains sesquilinear and antilinear forms: for $w, v \in H^1(\hat{\Omega}^o_m(\hat{\mu}^{\rm geo}_m))$, we define $\hat{a}^o_m(w, v; \hat{\mu}^{\rm coeff}_m; \hat{\mu}^{\rm geo}_m), \hat{f}^o_m(v; \hat{\mu}^{\rm coeff}_m; \hat{\mu}^{\rm geo}_m),\ \hat{\ell}^o_m(v; \hat{\mu}^{\rm coeff}_m; \hat{\mu}^{\rm geo}_m)$, $1 \leqslant m \leqslant M$, for coefficient parameters $\hat{\mu}^{\rm coeff}_m$ in associated coefficient parameter domains $\hat{\mathcal{D}}^{\rm coeff}_m \in \mathbb{R}^{\hat{P}^{\rm coeff}_m}$. We recall that for any domain $\mathcal{O}$ in $\mathbb{R}^d, H^1(\mathcal{O}) \equiv \{v \in L^2(\mathcal{O}) : \nabla v \in (L^2(\mathcal{O}))^d\}$, where $L^2(\mathcal{O}) \equiv \{v \text{ measurable over } \mathcal{O} : \int_{\mathcal{O}} |v|^2 \text{finite}\}$; note that we let $\overline{v}$ denote complex conjugate of v such that the complex modulus is then $|v| \equiv \sqrt{v\overline{v}}$. The forms $\hat{a}^o_m(w, v; \hat{\mu}^{\rm coeff}_m; \hat{\mu}^{\rm geo}_m)$ and $\hat{f}^o_m(v; \hat{\mu}^{rmcoeff}_m; \hat{\mu}^{\rm geo}_m)$ will define the PDE weak form, and $\hat{\ell}^o_m(v; \hat{\mu}^{\rm coeff}_m; \hat{\mu}^{\rm geo}_m)$ will define the output functional. Here we suppose that $w, v \in H^1\left(\hat{\Omega}^o_m(\hat{\mu}^{\rm geo}_m)\right)$ are *complex-valued* functions, and that $\hat{a}^o_m(w, v; \hat{\mu}^{\rm coeff}_m; \hat{\mu}^{\rm geo}_m)$ need not be a symmetric form – hence we generalize the formulation from [5]. We also define for future reference $\hat{\mathcal{D}}_m \equiv \hat{\mathcal{D}}^{\rm geo}_m \times \hat{\mathcal{D}}^{\rm coeff}_m, 1 \leqslant m \leqslant M; \hat{\mathcal{D}}_m$ is the parameter domain (both geometry and coefficient) for the mth archetype component. Also, we let $\hat{\mu}_m \equiv (\hat{\mu}^{\rm geo}_m, \hat{\mu}^{\rm coeff}_m) \in \hat{\mathcal{D}}_m$ denote the parameters for the mth archetype component.

We next introduce "instantiated component" domains, $\Omega^o_i(\mu^{\rm geo}_i) = \hat{\Omega}^o_{\mathcal{M}(i)}(\mu^{\rm geo}_i), 1 \leqslant i \leqslant I$, where $\mathcal{M}$ is a mapping from $\{1, \ldots, I\}$ (component instantiations) to $\{1, \ldots, M\}$ (component archetypes). The corresponding instantiated component port domains, also denoted more succinctly as "local ports," are thus given by $\gamma^o_{i,j}(\mu^{\rm geo}_i) = \hat{\gamma}^o_{\mathcal{M}(i),j}(\mu^{\rm geo}_i), 1 \leqslant j \leqslant n^\gamma_{\mathcal{M}(i)}$. Here $\mu^{\rm geo} \equiv (\mu^{\rm geo}_1, \ldots, \mu^{\rm geo}_I) \in \mathcal{D}^{\rm geo}$, where $\mathcal{D}^{\rm geo}$ is a subset of $\prod_{i=1}^I \hat{\mathcal{D}}^{\rm geo}_{\mathcal{M}(i)}$ which ensures that a geometric compatibility requirement is satisfied (as well as any problem–specific constraints on parameters). Note these instantiated components no longer bear the $^\wedge$ of the archetype components but retain the ^{o} associated with the physical domain. For future reference we also define $\mathcal{D}^{\rm coeff} \equiv \prod_{i=1}^I \hat{\mathcal{D}}^{\rm coeff}_{\mathcal{M}(i)}$. Also, we set $\mu \equiv (\mu^{\rm geo}, \mu^{\rm coeff}) \in \mathcal{D} \equiv \mathcal{D}^{\rm geo} \times \mathcal{D}^{\rm coeff}$.

We now consider a "system domain" $\Omega^o(\mu^{\rm geo})$ which is the union of instantiated component domains,

$$\overline{\Omega}^o(\mu^{\rm geo}) = \bigcup_{i=1}^I \overline{\Omega}^o_i(\mu^{\rm geo}_i).$$

We suppose that this system domain satisfies geometric compatibility conditions, and in particular that instantiated components intersect only over entire local ports – see [5] for more details.

Given the port connection requirements we can now define a set of global ports $\Gamma^o_p(\mu^{\rm geo}), 1 \leqslant p \leqslant n^\Gamma$: each global port is either the intersection (in fact, coincidence) of two local ports or a local port on $\partial\Omega^o(\mu^{\rm geo})$. We can summarize the port connections with index sets $\pi_p, 1 \leqslant p \leqslant n^\Gamma$, which for the case of a global port corresponding to (coincidence of) two local ports $\gamma^o_{i',j'}, \gamma^o_{i'',j''}$ takes the form $\pi_p = \{(i',j'),(i'',j'')\}$ and for the case of a global port corresponding to a single local port $\gamma_{i',j'}$ takes the form $\pi_p = \{(i',j')\}$. We can also define a local to global port index mapping $\mathcal{G}$ such that $\pi_p = \{(i',j'),(i'',j'')\}$ is equivalent to $p = \mathcal{G}_{i'}(j') = \mathcal{G}_{i''}(j'')$; this mapping is invertible for a given instantiated component such that $j' = \mathcal{G}^{-1}_{i'}(p)$ and $j'' = \mathcal{G}^{-1}_{i''}(p)$. We observe from the geometric compatibility condition that the sets $\pi_p, 1 \leqslant p \leqslant n^\Gamma$, and the mapping $\mathcal{G}$ do not depend on $\mu \in \mathcal{D}^{\rm geo}$.

We may now introduce a global function space $X^o(\mu^{\rm geo}) \equiv \{v^o \in H^1(\Omega^o(\mu^{\rm geo})) : v|_{\partial\Omega^o_D(\mu^{\rm geo})} = 0\}$ where $\partial\Omega^o_D(\mu^{\rm geo})$ represents the part of the system domain boundary over which we impose homogeneous Dirichlet conditions (note inhomogeneous Dirichlet conditions are readily treated by appropriate lifting functions). For simplicity, we assume that $\partial\Omega^o_D(\mu^{\rm geo})$ is the union of (at least one) entire instantiated component port domains. We endow $X^o(\mu^{\rm geo})$ with an inner product and induced norm,

$$(v,w)_{X^o(\mu^{\rm geo})}, \quad \|w\|_{X^o(\mu^{\rm geo})} = \sqrt{(v,w)_{X^o(\mu^{\rm geo})}}, \quad \forall v,w \in X^o(\mu^{\rm geo}), \tag{1}$$

which will be specified more precisely subsequently. It is then natural to form the system sesquilinear and antilinear forms, defined with respect to $X^o(\mu^{\rm geo})$, in terms of the corresponding archetype component forms introduced earlier. In particular, for all $w^o, v^o \in X^o(\mu^{\rm geo})$, we introduce

$$a^o(w^o, v^o; \mu^{\rm coeff}; \mu^{\rm geo}) = \sum_{i=1}^I \hat{a}^o_{\mathcal{M}(i)}\left(w^o|_{\Omega^o_i(\mu^{\rm geo}_i)}, v^o|_{\Omega^o_i(\mu^{\rm geo}_i)}; \mu^{\rm coeff}_i; \mu^{\rm geo}_i\right), \tag{2}$$

$$f^o(v^o; \mu^{\rm coeff}; \mu^{\rm geo}) = \sum_{i=1}^I \hat{f}^o_{\mathcal{M}(i)}\left(v^o|_{\Omega^o_i(\mu^{\rm geo}_i)}; \mu^{\rm coeff}_i; \mu^{\rm geo}_i\right), \tag{3}$$

$$\ell^o(v^o; \mu^{\rm coeff}; \mu^{\rm geo}) = \sum_{i=1}^I \hat{\ell}^o_{\mathcal{M}(i)}\left(v^o|_{\Omega^o_i(\mu^{\rm geo}_i)}; \mu^{\rm coeff}_i; \mu^{\rm geo}_i\right). \tag{4}$$

We may then introduce inf-sup and continuity constants for $\mu \in \mathcal{D}$,

$$\beta^o(\mu) \equiv \inf_{v^o \in X^o(\mu^{\rm geo})} \sup_{w^o \in X^o(\mu^{\rm geo})} \frac{|a^o(v^o, w^o; \mu^{\rm coeff}; \mu^{\rm geo})|}{\|v^o\|_{X^o(\mu^{\rm geo})}\|w^o\|_{X^o(\mu^{\rm geo})}}, \tag{5}$$

$$\gamma^{o,\rm cont}(\mu) \equiv \sup_{v^o \in X^o(\mu^{\rm geo})} \sup_{w^o \in X^o(\mu^{\rm geo})} \frac{|a^o(v^o, w^o; \mu^{\rm coeff}; \mu^{\rm geo})|}{\|v^o\|_{X^o(\mu^{\rm geo})}\|w^o\|_{X^o(\mu^{\rm geo})}}. \tag{6}$$

We assume that there exists $\alpha_0 > 0$ and finite $\gamma^{\rm cont}_0$ such that $\alpha^o(\mu) \geqslant \alpha_0$ and $\gamma^{o,\rm cont}(\mu) \leqslant \gamma^{\rm cont}_0$ for all $\mu \in \mathcal{D}$. We also assume that our antilinear functionals are bounded over $X^o(\mu^{\rm geo})$.

We may now state the system problem. Given $\mu \in \mathcal{D}$: find the field $u^o(\mu) \in X^o(\mu^{\rm geo})$ such that

$$a^o(u^o(\mu), v^o; \mu^{\rm coeff}; \mu^{\rm geo}) = f^o(v^o; \mu^{\rm coeff}; \mu^{\rm geo}), \quad \forall v^o \in X^o(\mu^{\rm geo}); \tag{7}$$

evaluate the system output of interest $s(\mu) = \ell^o(u^o(\mu); \mu^{\rm coeff}; \mu^{\rm geo})$, where in general $s(\mu) \in \mathbb{C}$. Under the assumptions on inf-sup