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A mixed formulation of mortar-based contact with friction

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ABSTRACT

A classical three-field mixed variational formulation of frictionless contact is extended to the frictional regime. The construction of the variational framework with respect to a curvilinear coordinate system naturally induces projected mortar counterparts of tangential kinetic and kinematic quantities while automatically satisfying incremental objectivity of the associated discrete penalty-regularized mortar constraints. Mixed contact variables that contribute to the boundary value problem are then obtained through unconstrained, lumped or constrained recovery approaches, complemented by Uzawa augmentations. Patch tests and surface locking studies are presented together with local and global quality monitors of the contact interactions in two- and three-dimensional settings at the infinitesimal and finite deformation regimes.

1. Introduction

The incorporation of friction into contact algorithms requires additional effort, in particular to ensure an objective integration of the evolution laws that describe irreversible interface mechanics but also in obtaining the associated algorithmic linearization to achieve an asymptotically quadratic convergence. The consideration of friction in the context of mortar approaches has developed in parallel with the frictionless case. Although the mortar discretization of the kinetic and kinematic contact variables are similar to the frictionless case, the description of the projected tangential kinematics is typically realized in terms of vector quantities and many studies are still limited to a two-dimensional setting. A vectorial description requires a careful construction of the update scheme for the tractions while the procedure is well-established in the continuum setting where the components with respect to a local curvilinear coordinate system are conveniently employed. Accordingly, contrary to normal contact, the way in which the tangential constraints are treated differs considerably in mortar approaches. In [1], the three-field mixed variational formulation foundation for mortar-based frictionless contact treatment was investigated. One goal of the present work is to construct a unified mixed variational treatment of normal and tangential contact constraints by extending the original idea of [2] for frictionless contact. The advantage of such a unification is that the algorithmic aspects of the complete tangential formulation emanate directly from a three-field statement, the only exception being the necessity of an independent mortar projection of the slip criterion. The advocated method lacks an intermediate surface to ensure a highly accurate numerical integration but it has all the remaining major ingredients of a mortar-based approach. Moreover, it offers alternatives to popular mortar schemes and provides numerical demonstrations of their viability. It is highlighted that a large class of algorithms associated with the classical node-to-segment approach and its variants are omitted from the present discussion where emphasis is strictly on mortar methods. The reader is referred to [3,4] for extensive references on alternative approaches, to [5] for a recent survey of numerical algorithms for contact problems.

Starting with applications in small deformation contact problems [6,7], it was shown that mortar methods satisfy the patch test while avoiding surface locking [8–10]. Additional investigations in the finite deformation regime with large sliding have further demonstrated the ability of the mortar method to successfully address problems that proved to be problematic for the classical node-to-segment schemes [11–18]. The incorporation of friction in mortar approaches goes back to the mathematical analysis of [19]. A finite element framework was subsequently investigated in [20] within a kinematically linearized two-dimensional framework. Based on parallel developments in domain decomposition [21], an extension to three-dimensional large deformation contact was introduced in [22]. Here, particular emphasis was given to the segmentation of the contact interface for an accurate evaluation of the contact integrals. Additionally, the construction of an incrementally objective slip definition was presented. Further investigations in a two-dimensional setting were presented in [23]. Two-dimensional studies on the possibilities for quadratic elements and integration

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without an intermediate surface were explored in [24]—see also [25] for three-dimensional implementations for quadratic elements with an intermediate surface. While the mentioned works incorporated a regularization of the normal and tangential constraints through a penalty method, a Lagrange multiplier approach was introduced in [26] in a two-dimensional setting and a three-dimensional framework with dual Lagrange multipliers was presented in [27]—see also [28] for two-dimensional investigations. Recent studies also introduced efficient semi-smooth Newton approaches, based on [29,30], for two-dimensional mechanical [31] and three-dimensional thermomechanical [32] problems. See [33,34] for further three-dimensional examples.

In order to construct a three-dimensional mixed formulation of mortar-based contact treatment with friction for large deformations, the three-field mixed variational formulation of [2] is extended to the tangential kinematics in Section 2 on the basis of the discussion in [1]. Starting from the continuum setting, a direct formulation with respect to a curvilinear coordinate system automatically ensures the objectivity of the stick and slip formulations. Constrained and lumped recovery approaches for the tangential interactions are developed and subsequently linked with an unconstrained formulation. Consistency statements with respect to Uzawa augmentations are discussed and subsequently the algorithmic linearization of the overall algorithm is presented in Section 3. Numerical investigations are presented in Section 4 where, in addition to comparing the local traction quality with analytical solutions, patch tests for tied non-flat surfaces as well as examples with significant evolutions of the contact interface are discussed.

Throughout the theoretical developments and numerical investigations, various details and observations have been omitted together with related references in order to minimize overlap with the presentation in [1]. In particular, the notation introduced therein is employed without restating definitions, i.e. the presentation is not entirely self-contained. However, a repetition of critical aspects has been incorporated to a minor extent.

2. Three-field mixed tangential contact treatment

2.1. Continuum formulation

The contact contribution to the weak form can be expressed as

$$\delta\mathcal{G}^c := \int_{\partial\mathcal{R}_0^c} (\delta\boldsymbol{x} - \delta\boldsymbol{y}) \cdot \boldsymbol{p}\, dA = -\int_{\partial\mathcal{R}_0^c} (\delta g_N p_N + \delta\xi^\alpha \tau_\alpha) dA, \tag{2.1}$$

under the standard assumption of an exact satisfaction of the impenetrability condition $g_N = 0$ to simplify the tangential contribution, with τ_α as the covariant components of the tangential traction [3,4]. Here, the convention is such that on the deformed configuration of the master surface the convected curvilinear coordinates ξ^α induce the covariant basis vectors $\boldsymbol{a}_\alpha := \frac{\partial \boldsymbol{y}}{\partial \xi^\alpha}$ which define the covariant metric components $a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$. The inverse (contravariant metric) components $a^{\alpha\beta}$, with $a^{\alpha\gamma} a_{\gamma\beta} = \delta^\alpha_\beta$ as the Kronecker delta, then define the contravariant basis vectors $\boldsymbol{a}^\alpha := a^{\alpha\beta}\boldsymbol{a}_\beta$ such that a generic vector $\boldsymbol{v}$ in the tangent space admits the representations $\boldsymbol{v} = v^\alpha \boldsymbol{a}_\alpha = v_\alpha \boldsymbol{a}^\alpha$.

Within a time/load-discretized setting with step index n, the incremental updates

$$g_T^\alpha := \xi^\alpha - \xi^{\alpha,n}, \quad p_{T\alpha} := \tau_\alpha - \tau_\alpha^n \tag{2.2}$$

are of interest where variables without a time/load index belong to $n+1$. For tangential contact, variational terms are identified in terms of the history variables and the incremental updates as

$$\delta\mathcal{G}_T^c = -\int_{\partial\mathcal{R}_0^c} \delta g_T^\alpha (\tau_\alpha^n + p_{T\alpha}) dA. \tag{2.3}$$

Here, the Coulomb slip criterion

$$\sigma(\boldsymbol{\tau}, p_N) := \|\boldsymbol{\tau}\| - \mu p_N \leqslant 0 \tag{2.4}$$

is assumed where μ is the (constant) friction coefficient and $\|\boldsymbol{\tau}\|^2 = \tau_\alpha a^{\alpha\beta}\tau_\beta$. During slip, the evolution of the projection coordinates is obtained from the objective statement

$$s_\alpha := \frac{\tau_\alpha}{\|\boldsymbol{\tau}\|} \rightarrow \dot{g}_T^\alpha = \dot{\lambda}\frac{\partial\sigma}{\partial\tau_\alpha} = \dot{\lambda} a^{\alpha\beta} s_\beta, \tag{2.5}$$

where $\dot{\lambda}$ is the consistency parameter, with which the tangential constraints can be stated as

$$\sigma \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \sigma\dot{\lambda} = 0. \tag{2.6}$$

2.2. Mixed formulation

Within a penalty regularization of the contact constraints, the tangential contact contribution to the weak form emanates, the latter assuming the stick state, from the variation of

$$\mathcal{G}_T^c = -\int_{\partial\mathcal{R}_0^c} \left(g_T^\alpha \tau_\alpha^n + \frac{\epsilon_T}{2} g_T^\alpha a_{\alpha\beta} g_T^\beta \right) dA, \tag{2.7}$$

such that $p_{T\alpha} = \epsilon_T a_{\alpha\beta} g_T^\beta$. Algorithmically, the continuum formulation leads to the update

$$\tau_\alpha = \tau_\alpha^n + \epsilon_T (a_{\alpha\beta} g_T^\beta - \Lambda s_\alpha), \tag{2.8}$$

where Λ, the time-discrete version of $\dot{\lambda}$, vanishes in the case of stick. In obtaining this update, the variation of $a_{\alpha\beta}$ is omitted from the weak form—see Section 3 for a discussion.

The approach of [2] for classical three-field mixed formulation of normal contact is now extended to tangential contact. The initial steps largely follow [1] and are only briefly addressed. Key difficulties associated with the kinetic quantities will be treated in detail. For this purpose, introducing *mixed tangential kinematic variables* γ_T^α, the following three-field mixed formulation in terms of $\{g_T^\alpha, \gamma_T^\alpha, p_{T\alpha}\}$ is introduced:

$$\mathcal{C}_T[g_T^\alpha, \gamma_T^\alpha, p_{T\alpha}] := -\frac{\epsilon_T}{2}\int_{\partial\mathcal{R}_0^c} \gamma_T^\alpha a_{\alpha\beta}\gamma_T^\beta dA + \int_{\partial\mathcal{R}_0^c} p_{T\alpha}(\gamma_T^\alpha - g_T^\alpha) dA - \int_{\partial\mathcal{R}_0^c} g_T^\alpha \tau_\alpha^n dA. \tag{2.9}$$

In this section, a trial stick stage is intrinsically assumed but not explicitly denoted for notational brevity. Unlike the normal part, the covariant metric must appear as an additional purely geometrical term in this functional. The variation of $\mathcal{C}_T$ delivers the tangential contribution $\delta\mathcal{G}_T^c$ to the linear momentum balance as well as the equalities

$$\int_{\partial\mathcal{R}_0^c} \delta p_{T\alpha}\gamma_T^\alpha dA = \int_{\partial\mathcal{R}_0^c} \delta p_{T\alpha} g_T^\alpha dA \tag{2.10}$$

and

$$\int_{\partial\mathcal{R}_0^c} \delta\gamma_T^\alpha p_{T\alpha} dA = \epsilon_T \int_{\partial\mathcal{R}_0^c} \delta\gamma_T^\alpha a_{\alpha\beta}\gamma_T^\beta dA. \tag{2.11}$$

In order to complement the normal formulation in the mortar setting, the tangential part needs to be defined in terms of mortar projections to the nodes. This is realized by admitting discretizations of the mixed variables which are inherited from the slave surface discretization via [11,22]

$$p_{T\alpha} = \sum_I N^I p_{T\alpha}^I, \quad \gamma_T^\alpha = \sum_I N^I \gamma_T^{\alpha,I}. \tag{2.12}$$