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A multiscale method with patch for the solution of stochastic partial differential equations with localized uncertainties

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ABSTRACT

We here propose a multiscale numerical method for the solution of stochastic parametric partial differential equations with localized uncertainties described with a finite number of random variables. It is based on a multiscale domain decomposition method that exploits the localized side of uncertainties and incidentally improves the conditioning of the problem by operating a separation of scales. An efficient iterative algorithm is proposed that requires the solution of a sequence of simple global problems at a macro scale, involving a deterministic operator, and local problems at a micro scale for which we have the possibility to use fine approximation spaces. Global and local problems are solved using tensor approximation methods allowing the representation of high dimensional stochastic parametric solutions. Convergence properties of these tensor based methods, which are closely related to spectral decompositions, benefit from the separation of scales. Different types of uncertainties are considered at the micro level. They may be associated with some variability in the operator or source terms, or even with some geometrical variability. In the latter case, specific reformulations of local problems using fictitious domain methods are introduced.

1. Introduction

Uncertainty quantification methods using functional approaches have emerged for the last two decades in computational engineering. Numerous developments have been realized for the propagation using functional approaches (see reviews [1–4]). In recent years, interest has grown for stochastic multiscale models and several numerical methods devoted to scale coupling have been extended to stochastic problems with global uncertainties (see e.g. [5–11]). Some of these methods, e.g. the Multiscale Finite Element Methods, draw their efficiency from assumptions as low perturbation hypothesis and yield results that are all the more precise as the scales are well separated. Other methods based on domain decomposition have also been proposed in [12–14] for handling multiscale stochastic problems and are again well suited when uncertainties occur in the whole domain.

Nevertheless, the propagation of uncertainties through multiscale stochastic models remains today a challenging issue for they give rise to high dimensional stochastic problems and this high dimensionality is thus to be handled genuinely. Moreover, monoscale numerical approaches clearly suffer from the complexity of multiscale solutions that present very high spectral content.

In the present work, we focus on multiscale problems with localized uncertainties (in medium property, source terms or geometry). In the presence of numerous localized sources of uncertainties, dedicated approaches have to be developed in order to handle the high dimensionality and complexity of associated multiscale models. At the deterministic level, dedicated methods have met the demand of coupling numerical models at different scales and some have been extended to stochastic models. Among these deterministic methods, one can distinguish the mono-model methods based on adaptive mesh or enrichment techniques [15–18] from the multi-model methods based on patches as the global–local iterative methods proposed in [19–23] or the bridging methods proposed in [24–26] or in [27] with the Arlequin method. The latter has been exploited in the stochastic framework for deterministic-stochastic coupling in [28,29] for a homogenization purpose.

We here propose a dedicated method based on a multiscale method with patches that exploits the localized side of uncertainties. It belongs to the class of the global–local iterative methods mentioned above. In its extension to the stochastic framework, an efficient iterative algorithm is proposed that requires the solution of a sequence of simple global problems at a macro scale, involving a deterministic operator, and local problems at a micro scale for which we have the possibility to use fine approximation spaces. In the meanwhile the separation of scales has the advantage of improving the conditioning of the problem. In order to

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address the high dimensionality that arises from these multiscale problems with numerous sources of uncertainties, the global and local problems are solved using tensor based approximation methods allowing the representation of high dimensional stochastic parametric solutions. Convergence properties of these tensor based methods, which are closely related to spectral decompositions, benefit from the separation of scales. Different formats of tensor representations can be exploited [30]. Here the classical canonical decompositions and the hierarchical canonical decompositions as in [31,32] are introduced, the latter ones giving very low ranks representations. The introduction of these decompositions within the proposed multiscale approach is a key point that makes the overall strategy very efficient.

Different types of uncertainties are considered at the micro level. They may be associated with some variability in the operator or source terms, or even with some geometrical uncertainty. In the latter case, specific reformulations of local problems using fictitious domain methods are introduced in order to formulate the problem on a tensor product space [33–35].

The paper is structured as follows. In Section 2, the model problem with localized variabilities is first presented. Then the global–local iterative algorithm is introduced in Section 3. Section 4 is dedicated to the approximate solutions of the global and local problems involved in the iterative algorithm: definition of approximation spaces and fictitious domain methods for the reformulation of the local problems when these present geometrical variabilities. Section 5 extends the method to the case of multiple patches with independent variabilities. The behavior of the global–local iterative algorithm is analyzed in Section 6 on the first numerical example with four patches and with no geometrical details. The convergence and robustness results of the global–local iterative algorithm proven in Section 3 are notably illustrated on this example. The influence of the sizes of the patches on the convergence of the algorithm is also analyzed. Tensor approximation methods are finally introduced in Section 7 for the solution of local and global problems in order to handle the high dimensionality. They are applied in Section 8 to a high dimensional problem which contains geometrical variabilities. This last illustration shows the relevance of the use of tensor approximation methods and in particular of hierarchical decompositions that provide very low-rank representations of local and global solutions.

2. Problem with localized variabilities

We consider a diffusion problem defined on a domain $\Omega \subset \mathbb{R}^d$:

$$
\begin{aligned}
&-\nabla \cdot (K \nabla u) = f \quad \text{on } \Omega,\\
&K \nabla u \cdot n = 0 \quad \text{on } \Gamma_N, \qquad (1)\\
&u = 0 \quad \text{on } \Gamma_D,
\end{aligned}
$$

with K a diffusion parameter, and Γ_D and Γ_N the Dirichlet and Neumann boundaries respectively. We denote by ξ a set of random parameters, with values in Ξ, modeling the uncertainties on the geometry, the source term and the diffusion coefficient. We denote by $(\Xi, \mathcal{B}, P_\xi)$ the associated probability space, where P_ξ is the probability law of ξ.

2.1. Function spaces

For a Hilbert space $\mathcal{H}$ equipped with an inner product norm $|\cdot|$, we denote by $\mathcal{H}^\Xi$ the Bochner space of square integrable functions defined on the measure space $(\Xi, \mathcal{B}, P_\xi)$ and with values in $\mathcal{H}$:

$$\mathcal{H}^\Xi = L^2_{P_\xi}(\Xi; \mathcal{H}) = \left\{ v : y \in \Xi \mapsto v(y) \in \mathcal{H};\ \mathbb{E}(|\ v(\xi)|^2) < +\infty \right\},$$

where $\mathbb{E}(\cdot)$ is the mathematical expectation defined by

$$\mathbb{E}(v) = \int_\Xi v(y)\, dP_\xi(y).$$

Bochner space $\mathcal{H}^\Xi$ is a Hilbert space when equipped with the following inner product norm

$$\|v\| = \mathbb{E}(|v(\xi)|^2)^{1/2}.$$

For $\mathcal{H} = \mathbb{R}$, we use the notations $\mathcal{S} := \mathbb{R}^\Xi = L^2_{P_\xi}(\Xi; \mathbb{R}) := L^2_{P_\xi}(\Xi)$. Note that $\mathcal{H}$ can be a random function space, *i.e.* dependent on ξ (e.g. when considering a space of functions defined on an uncertain domain). In the case where $\mathcal{H}$ is deterministic, the Bochner space can be identified with the tensor Hilbert space[1] $\mathcal{H} \otimes \mathcal{S}$:

$$\mathcal{H}^\Xi = L^2_{P_\xi}(\Xi; \mathcal{H}) \simeq \mathcal{H} \otimes \mathcal{S}.$$

2.2. Initial weak formulation of the problem

Let introduce the Hilbert space $\mathcal{V} = \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_D\}$ equipped with the inner product norm $|u|_\Omega = \left(\int_\Omega \nabla u \cdot \nabla u\right)^{1/2}$. Let $\mathcal{V}^\Xi = L^2_{P_\xi}(\Xi; \mathcal{V})$ be the Hilbert space equipped with the norm $\|\cdot\|_\Omega = \mathbb{E}(|\cdot|^2_\Omega)^{1/2}$. We introduce the classical weak formulation of problem (1):

$$u \in \mathcal{V}^\Xi, \quad a_\Omega(u, \delta u) = \ell_\Omega(\delta u) \quad \forall \delta u \in \mathcal{V}^\Xi, \qquad (2)$$

with

$$
\begin{aligned}
a_\Omega(u, \delta u) &= \mathbb{E}\left(\int_\Omega K \nabla u \cdot \nabla \delta u\right) = \int_\Xi \int_\Omega K \nabla u \cdot \nabla \delta u \, dP_\xi,\\
\ell_\Omega(\delta u) &= \mathbb{E}\left(\int_\Omega f \delta u\right) = \int_\Xi \int_\Omega f \delta u \, dP_\xi.
\end{aligned}
$$

We introduce the notation $\Omega * \Xi = \{(x, y) \in \mathbb{R}^d \times \Xi;\ x \in \Omega(y)\}$. Note that in the case of a deterministic domain Ω, we simply have $\Omega * \Xi = \Omega \times \Xi$. Problem (2) is well-posed if $f \in L^2(\Omega)^\Xi$ and if K is uniformly bounded and elliptic on $\Omega * \Xi$, *i.e.* there exist constants $K_{inf} > 0$ and $K^{sup} > 0$ such that we have almost everywhere on $\Omega * \Xi$

$$K_{inf}\ |\zeta|^2 \leqslant \zeta \cdot K(x, y)\zeta \leqslant K^{sup}\ |\zeta|^2 \quad \forall \zeta \in \mathbb{R}^d. \qquad (3)$$

2.3. Patch containing localized variabilities

We consider that the diffusion coefficient K, the source term f or the domain Ω are uncertain only on a part $\Lambda \subset \Omega$. Λ is called a patch. The boundary $\partial\Lambda$ of this patch contains the possible uncertain part of the boundary $\partial\Omega$. That means that Λ possibly depends on ξ and is such that

$$\Omega(\xi) = (\Omega \setminus \Lambda) \cup \Lambda(\xi),$$

with $\Omega \setminus \Lambda$ deterministic. We denote by $\Gamma = \partial(\Omega \setminus \Lambda) \cap \partial\Lambda$ the deterministic interface between $\Omega \setminus \Lambda$ and the patch Λ (see Fig. 1). We then consider that

$$K(x, \xi) = \begin{cases} K_0(x) & \text{for } x \in \Omega \setminus \Lambda \\ K(x, \xi) & \text{for } x \in \Lambda(\xi) \end{cases} \quad \text{and} \quad f(x, \xi) = \begin{cases} f_0(x) & \text{for } x \in \Omega \setminus \Lambda \\ f(x, \xi) & \text{for } x \in \Lambda(\xi) \end{cases}.$$

[1] Recall that an infinite dimensional tensor Hilbert space V is obtained by the completion with respect to some inner product norm of an algebraic tensor space ${}_a\otimes V_k = span\{\otimes_{k=1}^d v^k;\ v^k \in V_k\}$, where V_k are d vector spaces (we refer to [30], Chapter 4, for a detailed introduction to infinite dimensional tensor Banach spaces).