Comput. Methods Appl. Mech. Engrg. 254 (2013) 238–253

Contents lists available at SciVerse ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

Residual-based variational multiscale turbulence models for unstructured tetrahedral meshes

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ARTICLE INFO

Article history:
Received 8 March 2012
Received in revised form 31 August 2012
Accepted 20 September 2012
Available online 25 October 2012

Keywords:
Tetrahedral elements
Residual-based turbulence models
Variational multiscale method
Large eddy simulation
Stabilized finite elements
Bubble functions

ABSTRACT

This paper presents three-level residual-based turbulence models for the incompressible Navier–Stokes equations. Employing the variational multiscale (VMS) framework, the velocity and pressure fields are decomposed into two overlapping hierarchical scales, thereby leading to a system of coupled mixed field problems. The mixed problem at the fine scales is stabilized via a subsequent VMS application that results in a further decomposition of the fine-scale velocity field into level-I and level-II scales. The level-II scales are modeled using higher-order bubble functions that are then variationally embedded in the level-I formulation to stabilize it. The level-I problem is modeled via a second set of bubble functions that are linearly independent of the bubbles employed at level-II. Finally, the resulting level-I fine-scales are variationally embedded in the coarse-scale formulation. This yields a residual-based turbulence model for the larger or coarser-scales. A significant feature of the proposed method is that it results in a concurrent top-down and bottom up two-way nesting of the scales. In addition, the resulting turbulence model does not possess any embedded tunable parameters. Another attribute of the formulation is that the fine scales at every level are driven by the residuals of the Euler–Lagrange equations of the coarser scales at the preceding levels, thereby resulting in a method that is variationally consistent. Various algorithmic generalizations of the method are presented that lead to computationally economic residual-based turbulence models. The proposed telescopic depth in scales approach helps make these models accurate for low order tetrahedral and hexahedral elements, a feature that is facilitated by the higher-order bubble functions over element interiors and it results in an enhanced representation of the fine-scale terms modeling the fine viscous effects. From a computational perspective this method results in easy-to-implement equal-order pressure–velocity elements, and possesses the desirable p-refinement feature. Numerical performance of the method is assessed on turbulent channel flow problems at $Re = 395$ and $Re = 590$. Also presented are the results for turbulent SD-7003 airfoil at $Re = 60{,}000$ and comparison is made with the published experimental data and numerical results.

1. Introduction

Large eddy simulation (LES) is a numerical technique that resolves the larger features in the flow and models the effects of the smaller features. It is a powerful tool to study turbulent flows [38,24,14,9,33,26] and is computationally less expensive than Direct Numerical Simulation (DNS) [34] that tries to resolve all the scales in the problem.

This paper presents residual-based turbulence models that are derived via a nested application of the variational multiscale (VMS) ideas. The VMS framework assumes an overlapping additive decomposition of the unknown solution fields into coarse and fine-scale components and it was proposed by Hughes [17] as the basis for the development of stabilized methods. Variational multiscale ideas were extended to turbulence models [18–20,7,35] where coarse- and fine-scales were interpreted as the low and high wave numbers that were associated with the larger and smaller features in the flow. Some recent works that employ VMS ideas are [23,15,10,2,6,1,3,13,16] wherein larger structures in the flow are numerically resolved and finer structures are modeled, a feature that is common with the LES modeling ideas.

The present paper is an extension of our earlier work on the development of residual-based turbulence models [31] for the incompressible Navier–Stokes equations. In Ref. [31], we assumed an overlapping additive decomposition of only the velocity field, while the pressure field was not decomposed. Consequently, the fine-scale problem was a function of the coarse and fine velocity, and the coarse pressure field. A bubble functions based approach was adopted to extract the model for the spatio-temporal fine-scale velocity field that was then variationally embedded in the coarse-scale problem to yield the residual-based turbulence model.

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0045-7825/$ - see front matter © 2012 Published by Elsevier B.V.
http://dx.doi.org/10.1016/j.cma.2012.09.015

Numerical investigations with the method prompted us to add an element level divergence term for an improved representation of the local conservation of mass property. This formulation worked well for linear hexahedral elements and a variety of benchmark tests were carried out. An application of the method to tetrahedral element meshes manifested the inherent stiff response of the low order tetrahedra. This characteristic feature of the low order tetrahedral elements has been reported in the literature, primarily in the area of solid mechanics [39,37]. This prompted us to revisit our line of thought for the residual-based turbulence models and develop enhanced representations of fine scale fields that could compensate for the inherent stiffness of the lower order Lagrange elements. The models presented in [31] were based on an enhanced representation of the fine-scale velocity field and it was an extension of our earlier works on convection dominated flows [27,28,30,5] where fine-scale velocity was assumed piece-wise constant in time. In other words, a more general representation of the fine scale velocity field had resulted in a refined fine scale model which had in turn resulted in a residual-based turbulence model that worked well for the hexahedral elements.

Our motivation in this work is to further develop the refined representations of the fine-scale velocity and pressure fields that can compensate for the inherently stiff response of the low-order tetrahedral elements. We perform a nested and hierarchical application of the VMS method, and from the onset, assume a multiscale decomposition of both the velocity and pressure fields. Consequently, the problem that governs fine scales is also a mixed field problem, and therefore it needs to be stabilized if arbitrary interpolation functions are to be used for the fine-scale velocity and pressure fields. To accomplish this, we perform another application of the VMS ideas, and further decompose the fine-scale velocity into two overlapping components termed as fine-scales level-I and level-II. The goal of level-II scales is to provide VMS based stabilization to the system of equations governing level-I scales. Subsequent variational projection of the fine-scales that are obtained from the stabilized level-I equations, onto the coarse scale problem, yields the desired turbulence model. While the coarse scales are interpolated using standard shape functions, level-I and level-II scales are modeled employing bubble functions. Specifically, the presence of fine-scale pressure field allows us to consistently derive terms that are analogous to the so-called "*div*-stabilization" term, which help improve the conservation of mass property in the model. We describe this aspect further in Section 3.1.

The remaining part of the paper is organized as follows. The Navier–Stokes equations and their weak formulation are presented in Section 2. In Section 3 we derive the three-scale residual-based turbulence model employing the variational multiscale framework. In Section 3 the detailed development of the fine-scale model is presented and the various modeling simplifications that are taken into consideration are discussed. Section 4 presents several numerical tests to show the accuracy of the formulation for unstructured tetrahedral meshes. Section 4.1 considers a turbulent channel flow which is a classical benchmark problem for validating turbulence models. We compare our results with reference DNS solutions and with other LES models published in the literature [2,31]. We also propose some algorithmic simplifications for computational economy that yield some variants of the underlying model, and study the effects of these simplifications on the computed solution. Specifically we investigate the effects of diagonalization of the second order tensor $\boldsymbol{\tau}$ that arises in the derivation of the turbulence models. The effects of the modeling simplifications related to the temporal domain are also investigated. In Section 4.2 we study flow around an airfoil at Reynolds numbers 60,000 to show the applicability of our method to more complex problems. Conclusions are drawn in Section 5.

2. The incompressible Navier–Stokes equations

Let $\Omega \subset \mathbb{R}^3$ be a connected, open, bounded domain with piecewise smooth boundary Γ. Let $\boldsymbol{v} : \Omega \times]0,T[\to \mathbb{R}^3$ be the velocity field and $p : \Omega \times]0,T[\to \mathbb{R}$ be the kinematic pressure field. The incompressible Navier–Stokes equations can be written in the space–time domain $\Omega \times]0,T[$ as follows:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{v}) - 2\nu \nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) + \nabla p = f \quad \text{in } \Omega \tag{1}$$

$$\nabla \cdot \boldsymbol{v} = 0 \quad \text{in } \Omega \tag{2}$$

$$\boldsymbol{v} = \boldsymbol{g} \quad \text{on } \Gamma \tag{3}$$

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0 \quad \text{in } \Omega \tag{4}$$

where $\boldsymbol{f} : \Omega \times]0,T[\to \mathbb{R}^3$ is the body force (per unit of mass), $\boldsymbol{v} > 0$ is the kinematic viscosity (assumed constant), v_0 is the initial condition for the velocity field which satisfies the condition that $\nabla \cdot \boldsymbol{v}_0 = 0$, $\boldsymbol{g}$ represents the Dirichlet boundary conditions, and $\otimes$ denotes tensor product. The strain-rate tensor is defined as $\boldsymbol{\varepsilon}(\boldsymbol{v}) = \nabla^s \boldsymbol{v} = [\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T]/2$. Eq. (1) is the momentum balance equation; Eq. (2) enforces the incompressibility constraint; Eq. (3) is the Dirichlet boundary condition; and Eq. (4) is the initial condition.

Let $\boldsymbol{w}(x) \in \mathcal{v} = (H_0^1(\Omega))^3$ and $q(\boldsymbol{x}) \in \mathcal{Q} = C^0(\Omega) \cap L^2(\Omega)$ represent the weighting functions for the velocity and pressure fields, respectively. The appropriate spaces of functions for the velocity and pressure trial solutions are the corresponding time-dependent spaces $\mathcal{S}$ and $\mathcal{P}$ that satisfy the initial and boundary conditions. The standard weak form of the problem defined in (1)–(4) is: Find $\boldsymbol{v}(\boldsymbol{x}, t) \in \mathcal{S}$ and $p(x, t) \in \mathcal{P}$ such that for all $\boldsymbol{w}(\boldsymbol{x}) \in \mathcal{v}$ and $q(x) \in \mathcal{Q}$,

$$\left(\boldsymbol{w}, \frac{\partial \boldsymbol{v}}{\partial t}\right) - (\nabla \boldsymbol{w}, \boldsymbol{v} \otimes \boldsymbol{v}) + (\nabla^s \boldsymbol{w}, 2\nu \nabla^s \boldsymbol{v}) - (\nabla \bullet \boldsymbol{w}, p) = (\boldsymbol{w}, \boldsymbol{f}) \tag{5}$$

$$(q, \nabla \cdot \boldsymbol{v}) = 0 \tag{6}$$

where $(\bullet, \bullet) = \int_\Omega (\bullet) d\Omega$ is the $L^2(\Omega)$-inner product. Eqs. (5) and (6) imply weak satisfaction of the momentum balance equations and the continuity equation, in addition to the initial condition.

Remark: The shape functions of the linear tetrahedral element are incomplete Lagrange polynomials that do not possess the cross terms. As a result, the cross derivative are zero, and therefore cross terms in the weak form (5) and (6) are not accurately represented. This makes tetrahedral elements behave stiff as compared to the hexahedral elements that are complete Lagrange polynomials and therefore possess the cross terms.

Remark: The objective in Section 3 is to develop a formulation for tetrahedral elements that is able to model the smallest features in the solution of (5) and (6) via a better representation of fine scale viscous terms, thereby overcoming the limitations of the standard tetrahedral elements.

Remark: Use of tetrahedral elements in computational mechanics has primarily been attempted in the domain of solid and structural mechanics [39,37]. The stiff behavior of linear tetrahedral elements was recognized early on because of the constant strain nature of the fields produced by it. In the case of fluid mechanics where viscous term plays a significant role, the lack of cross terms in the expansion of the interpolation functions over the parent element domain becomes a serious impediment in an accurate representation of the viscous term. It is important to note that linear hexahedral elements and the 10-noded tetrahedral elements contain cross terms in the parent domain that provide them the flexibility to better represent the viscous terms.