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Stabilized finite elements for transient flow problems on varying spatial meshes ☆

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ABSTRACT

Changing the spatial mesh in transient flow computations may negatively affect the pressure on the new mesh due to the fact that the interpolated or L^2-projected velocities usually violate the divergence constraint on the new mesh. It is proven that this pressure perturbation scales as k^{-1} when k denotes the time step. Hence, this phenomenon becomes increasingly relevant for small time steps. This is even more important due to the fact that this phenomena occurs independently whether the discrete scheme is inf-sup stable or not. In order to solve this problem, a divergence free projection should be applied instead of a simple interpolation or L^2-projection of the velocities. For inf–sup stable finite elements, a recent published analysis shows how such a projection should be performed. For non inf-sup stable finite element pairs with stabilization techniques, as for instance equal-order elements, such an analysis is still missing. In this work, we tackle this problem, present a possible algorithm and prove bounds of the pressure in the linear Stokes case. The type of pressure stabilization is very general and includes the interior penalty method, local projection techniques and others.

1. Introduction

Adaptive meshing for non-steady problems attract notice as an efficient solution method for the numerical solution of partial differential equations. In flow problems (Stokes, Navier–Stokes, Oseen), the behavior of the pressure may be negatively affected when the mesh is changing from one time step to the next one. This is due to the fact, that the interpolated or L^2-projected velocity of the previous time step onto the new mesh is in general not discrete divergence-free. As shown recently by Besier and Wollner [1], this pressure perturbation is of order k^{-1} if k is the time step. Moreover, this effect is independent of the type of spatial discretization and present in most of the time discretizations. In particular, this phenomenon occurs for inf-sup stable finite elements as well as for non inf-sup stable finite elements. A remedy of such spurious pressure behavior is a 'divergence-free' projection of the old velocities onto the new mesh. For inf-sup stable elements it was proven in [1] that this scheme avoids the pressure peaks. For equal-order finite elements such a result is still unknown.

In this work, we focus on the 'divergence-free' projection of *stabilized* finite element schemes. This is reasonable because of the following two reasons: (i) many finite element methods (first of all the attractive equal-order schemes) are not inf-sup stable, and (ii) even stable Stokes elements are not stable for the 'divergence-free' projections of Darcy systems, see Mardal et al. [2]. We show that the algorithm needs additional terms and we prove stability of the pressure independent of the time step k. These results apply to several types of stabilized schemes, e.g. to Brezzi–Pitkaränta [3], local projection (LPS) [4], and interior penalty methods (IP) [5].

1.1. Notations

Let $\Omega\subset\mathbb{R}^d, d\in\{2,3\}$, be a bounded domain with polyhedral boundary. For the analysis we will later on assume Ω to be convex. $I=(0,T)$ is a time interval. By $(\cdot,\cdot)$ and $\|\cdot\|$ we denote the $L^2(\Omega)$ scalar product and norm, respectively. $L^2_0(\Omega)$ is the subspace of $L^2(\Omega)$ functions with zero integral mean value. Vector-valued quantities are printed in bold.

1.2. Navier–Stokes equations

We consider the non-steady Navier–Stokes equations in dimensionless form: find a velocity field $v: I\times\Omega\to\mathbb{R}^d$ and a pressure field $p: I\times\Omega\to\mathbb{R}$ such that

$$\partial_t\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}-\nu\Delta\mathbf{v}+\nabla p=\mathbf{f}\quad\text{in }\Omega,\tag{1}$$
$$\operatorname{div}\mathbf{v}=0\quad\text{in }\Omega,\tag{2}$$
$$\mathbf{v}=\mathbf{0}\quad\text{on }\partial\Omega,\tag{3}$$
$$\mathbf{v}|_{t=0}=\mathbf{v}_0\quad\text{in }\Omega,\tag{4}$$

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where $\nu > 0$ denotes a positive constant, $f : I \times \Omega \to \mathbb{R}^d$ and $\mathbf{v}_0 : \Omega \to \mathbb{R}^d$ are given functions. For ease of presentation, we restrict ourselves to homogeneous Dirichlet boundary conditions.

1.3. Backward Euler scheme in time

For the numerical computation of solutions we have to discretize in time and space. For simplicity, we will use the first order backward Euler scheme in time.

We consider m equidistant time steps of length $k := T/m$, $0 = t_0 < t_1 < \ldots < t_m = T$, with $t_i = ik$. In order to formulate the backward Euler scheme with $\mathbf{v}^{(i)} \approx \mathbf{v}(t_i)$ and $p^{(i)} \approx p(t_i)$ in variational formulation we introduce the variational spaces $\mathbf{V} := H_0^1(\Omega)^d$ and $Q := L_0^2(\Omega)$. We multiply the Eqs. (1) and (2) by test functions $\psi \in \mathbf{V}$ and $\chi \in Q$, respectively, and integrate certain terms by parts. This leads to the semi-discrete variational formulation: Find the pair $(\mathbf{v}^{(i)}, p^{(i)}) \in \mathbf{X} := \mathbf{V} \times Q$ such that for all $\psi \in \mathbf{V}$ and all $\chi \in Q$ it holds

$$\frac{1}{k}(\mathbf{v}^{(i)}, \psi) + ((\mathbf{v}^{(i)} \cdot \nabla)\mathbf{v}^{(i)}, \psi) + (\nu \nabla \mathbf{v}^{(i)}, \nabla \psi) - (p^{(i)}, \operatorname{div} \psi)$$
$$= \frac{1}{k}(\mathbf{v}^{(i-1)}, \psi) + (\mathbf{f}(t_i), \psi) \tag{5}$$
$$(\operatorname{div} \mathbf{v}_i, \chi) = 0. \tag{6}$$

1.4. Discussion of the underlying problem

For the numerical solution of the variational problem stated above, the infinite dimensional space $\mathbf{X}$ has to be replaced by a finite dimensional space $\mathbf{X}_H$. Here, H denotes the mesh size corresponding to a mesh $\mathcal{T}_H$. For conforming finite elements it holds $\mathbf{X}_H \subset \mathbf{X}$. The exact requirements are stated below. Let $(\mathbf{v}_H^{(i)}, p_H^{(i)}) \in X_H$ be the solution of the discrete analogon of (5), (6). After changing the mesh to another one, denoted by $\mathcal{T}_h$, by a partial or global mesh refinement of $\mathcal{T}_H$, let $\mathbf{v}_h^{(i)}$ be the L^2-projected velocity, i.e.

$$(\mathbf{v}_h^{(i)}, \psi) = (\mathbf{v}_H^{(i)}, \psi) \quad \forall \psi \in \mathbf{V}_h.$$

This projected velocity is needed in the right hand side to compute the discrete solution $(\mathbf{v}_h^{(i+1)}, p_h^{(i+1)}) \in \mathbf{X}_h$ of the next time step. The problem is that if $\mathbf{V}_h \not\subset \mathbf{V}_H$ (which is the case, e.g., after mesh refinement) the L^2-projected velocity is not divergence free with respect to the new test functions, i.e.

$$(\operatorname{div} \mathbf{v}_h^{(i)}, \chi) \neq 0$$

for certain $\chi \in Q_h$. In [1] it was shown that as a consequence, the new pressure $p_h^{(i+1)}$ becomes unbounded for vanishing time steps:

$$\|p_h^{(i+1)}\| \geqslant \frac{C}{k}$$

with a positive constant $C > 0$. To cure this pressure perturbations, divergence-free projections are proposed in [1] to define $\mathbf{v}_h^{(i)}$ instead of L^2-projection or interpolation. There this approach is analyzed for inf-sup stable finite element pairs $\mathbf{V}_h \times Q_h$. However, for not inf-sup stable finite element pairs, as e.g. equal-order elements, the situation is not clear yet. The reason is the additional stabilization terms arising in the discrete divergence equation. In this work, we will close this gap and show theoretically and by means of numerical examples how to proceed in this case.

1.5. Structure of the work

The material in this work is structured as follows. In the next section we define the discrete finite element spaces and introduce a general form of stabilization techniques to treat the absence of a discrete inf-sup condition. We formulate certain conditions for the stabilization which are needed later in the analysis. Several types of stabilization techniques are covered by these assumptions. In Section 3, the divergence free projection is formulated and several bounds are derived. These are needed in the subsequent sections. For this projection, stabilization techniques may be needed as well. In Section 4, we show that the resulting discrete system delivers in the case of the Stokes system bounded pressures. Numerical examples in Section 5 illustrate the behavior of the resulting algorithm.

2. Finite element approximation in space

For discretization in space we use finite elements. Let $\mathcal{T}_h$ be a shape-regular, admissible decomposition of Ω into either triangles or quadrilaterals for $d = 2$ or either simplices or hexahedra for $d = 3$. The outer diameter of a cell $K \in \mathcal{T}_h$ will be denoted by h_K and the maximal and minimal mesh sizes are defined as $h_{max} := \max\{h_K : K \in \mathcal{T}_h\}$ and $h_{min} := \min\{h_K : K \in \mathcal{T}_h\}$, respectively. For a family of quasi-uniform meshes it holds $h_{max} \leqslant c h_{min}$ with a mesh independent constant c. Let $\widehat{K}$ denote the reference element, $F_K : \widehat{K} \to K$ an isoparametric transformation to the physical cell K, and $\mathbb{P}_r(\widehat{K})$ the space of all polynomials on $\widehat{K}$ with total degree (in the case of simplices) or maximal degree $r \geqslant 0$ in each coordinate direction (in the case of quadrilaterals/hexahedrons). We will use the H^1-conforming finite element space

$$\mathbb{P}_r := \{ v_h \in H^1(\Omega) : v_h|_K \circ F_K \in \mathbb{P}_r(\widehat{K}) \quad \forall K \in \mathcal{T}_h \}.$$

The finite element spaces are $\mathbf{V}_h := (\mathbb{P}_r)^d \cap \mathbf{V}$ for the velocities, and $Q_h := \mathbb{P}_s \cap Q$ for the pressure with $1 \leqslant s \leqslant r$. The set of cell edges (faces in 3D) is denoted by $\mathcal{E}_h$.

2.1. Stabilized variational formulation

It is well-known that for equal-order elements, $s = r$, the Galerkin formulation of the Navier–Stokes system is not stable, i.e. the existence of a unique discrete pressure is not ensured. In this case, stabilization terms have to be added in the discrete formulation. There are many possibilities of such stabilization terms. Every method has its particular advantages and disadvantages. Instead of limiting to one particular technique, we will treat stabilization techniques in general. The corresponding additional semi-linear form will be denoted by $S_h(\mathbf{v}, p; \psi, \chi)$. Together with the semi-linear form of the Galerkin part with skew-symmetric convective part

$$A(\mathbf{w}; \mathbf{v}, p; \psi, \chi) := \frac{1}{k}(\mathbf{v}, \psi) + \frac{1}{2}(((\mathbf{w} \cdot \nabla)\mathbf{v}, \psi) - (\mathbf{v}, (\mathbf{w} \cdot \nabla)\psi))$$
$$+ (\nu \nabla \mathbf{v}, \nabla \psi) - (p, \operatorname{div} \psi) + (\operatorname{div} \mathbf{v}, \chi)$$

and the stabilized semi-linear form

$$A_h(\mathbf{v}; \mathbf{v}, p; \psi, \chi) := A(\mathbf{v}; \mathbf{v}, p; \psi, \chi) + S_h(\mathbf{v}; \mathbf{v}, p; \psi, \chi)$$

one backward Euler step reads

$$A_h(\mathbf{v}_h^{(i)}; \mathbf{v}_h^{(i)}, p_h^{(i)}; \psi, \chi) = \frac{1}{k}(\mathbf{v}_h^{(i-1)}, \psi) + (\mathbf{f}(t_i), \psi) \quad \forall (\psi, \chi) \in \mathbf{X}_h,$$

where $\mathbf{v}_h^{(i-1)} \in \mathbf{V}_h$ is the discrete velocity of the previous time step. Note, that we allow for $S_h \equiv 0$ in the case of inf-sup stable pairs $\mathbf{V}_h \times Q_h$ and without convection stabilization.

2.2. Considered methods of stabilization

As mentioned earlier, we allow for several types of methods. However, for the analysis we need certain properties of $S_h(\cdot, \cdot)$.