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A correction technique for the dispersive effects of mass lumping for transport problems ☆

Jean-Luc Guermond [a,*,1], Richard Pasquetti [b]

[a] Department of Mathematics, Texas A&M University 3368 TAMU, College Station, TX 77843, USA
[b] Lab. J.A. Dieudonné, UMR CNRS 7351, University of Nice-Sophia Antipolis, 06108 Nice, France

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ABSTRACT

This paper addresses the well-known dispersion effect that mass lumping induces when solving transport-like equations. A simple anti-dispersion technique based on the lumped mass matrix is proposed. The method does not require any non-trivial matrix inversion and has the same anti-dispersive effects as the consistent mass matrix. A novel quasi-lumping technique for $\mathbb{P}_2$ finite elements is introduced. Higher-order extensions of the method are also discussed.

1. Introduction

Lumping the mass matrix is a routine procedure in the finite element community when solving the heat equation, the wave equation and the time-dependent transport equation. This technique consists of replacing the consistent mass matrix by a diagonal surrogate usually referred to as the lumped mass matrix. This process avoids having to invoke sophisticated linear algebra arguments to invert the consistent mass matrix at each time step. The mantra in the literature dedicated to mass lumping is that mass lumping produces explicit algorithms for the transport and the wave equations that are algebra-free.

The lumped mass matrix is generally obtained by using a quadrature formula instead of exact integration. It is usually believed that lumping is a benign operation since it does not affect the overall accuracy of the method provided the quadrature is accurate enough. For instance, it is known that using quadrature formulas that are exact for $\mathbb{P}_{2k-2}$ polynomials is sufficient to preserve the overall accuracy of the Galerkin method when solving the wave equation or some eigenvalue problems on simplex meshes [1,7,12,11,20]. Although it is convenient numerically, it is well-known that lumping the mass matrix induces dispersion errors that have adverse effects when solving transport-like equations, see e.g. [5,6,14,22]. The objectives of the present work are as follows:

(i) We propose a simple correction technique based on the lumped mass matrix that does not involve sophisticated linear algebra and that has the same anti-dispersive effects as the consistent mass matrix. Although this correction technique relies on a matrix series, we show theoretically and numerically that only considering the first term in this series is enough to correct the dominating dispersion error.
(ii) We introduce a novel quasi-lumping technique for $\mathbb{P}_2$ finite elements, where the new $\mathbb{P}_2$ quasi-lumped mass matrix is triangular. We show also that the proposed mass correction technique is efficient when using this $\mathbb{P}_2$ quasi-lumped mass matrix.
(iii) We investigate higher-order extensions of the correction method and demonstrate satisfactory results for the $\mathbb{P}_3$ approximation.

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* Corresponding author.
E-mail address: guermond@math.tamu.edu (J.-L. Guermond).
[1] On leave from CNRS, France.

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To the best of our knowledge, the correction techniqud the quasi-lumping technique for $\mathbb{P}_2$ finite elements are original.

This paper is organized as follows. The anti-dispersive effects of the consistent $\mathbb{P}_1$ mass matrix on the transport equation are analyzed in Section 2. We focus in this section on the linear transport equation in one space dimension. Most of the material therein is standard. A mass correction technique based on the lumped mass matrix is presented in Section 3. The method has the same algebraic complexity as when using the lumped mass matrix. It is also proved for $\mathbb{P}_1$ elements in one space dimension that using one correction term only is enough to obtain the same anti-dispersive effect as when using the consistent mass matrix. The mass correction method is further evaluated numerically in two space dimension on $\mathbb{P}_1$ finite elements in Section 4. A new $\mathbb{P}_2$ quasi-lumping technique is introduced in Section 5. To the best of our knowledge, the $\mathbb{P}_2$ quasi-lumping technique presented in Section 5.3 and Section 5.4 and the mass correction technique introduced in Section 3 are original. Higher-order suboptimal variants of the method are considered in Section 6. Conclusions are reported in Section 7.

2. One-dimensional heuristics

The objective of this section is to analyze in details the effects of mass lumping in one space dimension for the linear transport equation using piece-wise linear finite elements. The material herein is certainly not new, see e.g. [6,14,17,22], but it is useful to comprehend the rest of the paper. Let us consider the following one-dimensional transport equation in the domain $\Omega = (a,b)$

$$\partial_t \mathrm{u} + \beta \partial_x \mathrm{u} = 0, \quad \mathrm{u}(x,0) = \mathrm{u}_0(x), \quad (x,t) \in (a,b)\times\mathbb{R}_+, \tag{2.1}$$

equipped with periodic boundary conditions. The velocity field β is assumed to be constant to simplify the presentation.

2.1. Galerkin linear approximation

Let us partition $\Omega = (a,b)$ into N intervals $[x_i, x_{i+1}]$, $i = 0,\dots,N-1$. Let $h_{i+\frac{1}{2}} := |x_{i+1} - x_i|$ be the diameter of the cell $[x_i, x_{i+1}]$. We introduce the family $\{\psi_0,\dots,\psi_N\}$ composed of continuous and piecewise linear Lagrange functions associated with the nodes $\{x_0,\dots,x_N\}$, and we define the $\mathbb{P}_1$ finite element space

$$\begin{aligned} X_h &= \left\{ v \in \mathcal{C}^0_{\#}(\overline{\Omega};R),\ v|_{[x_i,x_{i+1}]} \in \mathbb{P}_1,\ i = 0,\dots,N-1 \right\} \\ &= \mathrm{span}(\psi_0,\dots,\psi_N), \end{aligned} \tag{2.2}$$

where $\mathcal{C}^0_{\#}(\overline{\Omega};R)$ denotes the space of the real-valued functions that are periodic and continuous over $\overline{\Omega}$. Let u_0 be a reasonable approximation of u_0, say the Lagrange interpolate or L^2-projection thereof. An approximate solution to (2.1) is constructed by means of the Galerkin technique. We seek $u \in \mathcal{C}^1((0,T);X_h)$ so that $u(0) = u_0$ and

$$\int_\Omega (\partial_t u + \beta \partial_x u) v \mathrm{d}x = 0, \quad \forall v \in X_h. \tag{2.3}$$

The approximate solution $u(x,t)$ is expanded with respect to the basis $\{\psi_0,\dots,\psi_N\}$ as follows: $u(x,t) = \sum_{j=0}^N u_j(t)\psi_j(x)$. A system of ordinary differential equations is obtained by testing (2.3) with the members of the basis $\{\psi_0,\dots,\psi_N\}$.

Upon testing (2.3) with ψ_i, $i = 0,\dots,N$, the term involving the time derivative gives

$$\int_\Omega \partial_t u(x,t)\psi_i(x)\mathrm{d}x = \sum_{j=0}^N M_{ij}\partial_t u_j(t), \tag{2.4}$$

where the coefficients of the so-called mass matrix are

$$M_{ij} := \int_{x_{i-1}}^{x_{i+1}} \psi_i(x)\psi_j(x)\mathrm{d}x = \begin{cases} \frac{1}{6}h_{i+\frac{1}{2}} & \text{if } j = i\pm 1 \\ \frac{1}{3}\left(h_{i-\frac{1}{2}} + h_{i+\frac{1}{2}}\right) & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

with the convention that $h_{-\frac{1}{2}} = h_{N-\frac{1}{2}}$ and $h_{N+\frac{1}{2}} = h_{\frac{1}{2}}$. The transport term in (2.3) is handled as follows:

$$\begin{aligned} \int_\Omega \psi_i(x)\beta\partial_x u(x,t)\mathrm{d}x &= -\int_{x_i-1}^{x_{i+1}} \beta u(x,t)\partial_x\psi_i(x)\mathrm{d}x \\ &= \frac{\beta}{2}(u_{i+1}(t) + u_i(t)) - \frac{\beta}{2}(u_i(t) + u_{i-1}(t)), \end{aligned} \tag{2.6}$$

giving

$$\int_\Omega \psi_i(x)\beta\partial_x u(x,t)\mathrm{d}x = \beta\frac{1}{2}(u_{i+1}(t) - u_{i-1}(t)), \tag{2.7}$$

with the convention $u_{-1}(t) = u_{N-1}(t)$ and $u_{N+1}(t) = u_1(t)$.

Recalling that we are looking for a periodic solution, the above computation shows that the vector $(u_0(t),\dots,u_{N-1}(t))^T \in \mathbb{R}^N$ solves the following system of ordinary differential equations:

$$\sum_{j=i-1}^{i+1} M_{ij}\partial_t u_j(t) = -\beta\frac{1}{2}(u_{i+1}(t) - u_{i-1}(t)), \quad 0 \leqslant i,j < N, \tag{2.8}$$

where $u_N(t) = u_0(t)$ and $u_{-1}(t) = u_{N-1}(t)$. The above system can be written in matrix form as follows:

$$M\partial_t U(t) = F(U(t)), \tag{2.9}$$

with $U(t) := (u_0(t),\dots,u_{N-1}(t))^T$, and the entries of F are defined by $F_i(U) := -\beta\frac{1}{2}(u_{i+1} - u_{i-1}), 0 \leqslant i < N$, and where M is the consistent mass matrix defined in (2.5) taking into account the periodicity in the first and last lines.

2.2. Dispersion and mass lumping

It is common in the literature to approximate (2.9) in time by means of explicit time stepping. To avoid having to solve linear systems involving the mass matrix at each time step, it also common to simplify (2.8) by lumping the mass matrix. Mass lumping can be shown in one space dimension to be equivalent to approximate the consistent mass matrix by using the following trapezoidal quadrature rule:

$$\int_r^s f(x)\mathrm{d}x \approx (s-r)\frac{1}{2}(f(r) + f(s)). \tag{2.10}$$

This quadrature is exact for linear polynomials. Using this quadrature, the mass matrix coefficients can be approximated as follows:

$$\int_{x_{i-1}}^{x_{i+1}} \psi_i(x)\psi_j(x)\mathrm{d}x \approx \frac{1}{2}\left(h_{i-\frac{1}{2}} + h_{i+\frac{1}{2}}\right)\delta_{ij} =: \overline{M}_{ij}, \tag{2.11}$$

where δ_{ij} is the Kronecker symbol. The so-called lumped mass matrix $\overline{M}$ thus computed is diagonal. Upon denoting $\overline{h}_i := \frac{1}{2}\left(h_{i-\frac{1}{2}} + h_{i+\frac{1}{2}}\right)$ and replacing the consistent mass matrix by the lumped mass matrix, we obtain a new approximate form of transport equation as follows:

$$\partial_t \tilde{u}_i(t) + \beta\frac{\tilde{u}_{i+1} - \tilde{u}_{i-1}}{2\overline{h}_i} = 0. \tag{2.12}$$

The approximation thus constructed is second-order accurate. More precisely, the consistency error of (2.12) is characterized by the following.

Proposition 2.1. *Provided the mesh is uniform, of mesh size h, the dominating term in the consistency error of (2.12) at the grid points $\{x_i\}_{0\leqslant i\leqslant N}$ is dispersive and is equal to $\beta\frac{h^2}{6}\partial_{xxx}\mathrm{u}(x_i,t)$.*