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On the spatial formulation of discontinuous Galerkin methods for finite elastoplasticity ☆

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ABSTRACT

In this paper, we present a consistent spatial formulation for discontinuous Galerkin (DG) methods applied to solid mechanics problems with finite deformation. This spatial formulation provides a general, accurate, and efficient DG finite element computational framework for modeling nonlinear solid mechanics problems. To obtain a consistent formulation, we employ the Incomplete Interior Penalty Galerkin (IIPG) method. Another requirement for achieving a fast convergence rate for Newton's iterations is the consistent formulation of material integrators. We show that material integrators that are well developed and tested in continuous Galerkin (CG) methods can be fully exploited for DG methods by additionally performing stress returning on element interfaces. Finally, for problems with pressure or follower loading, stiffness contributed from loaded surfaces must also be consistently incorporated. In this work, we propose the Truesdell objective stress rate for both hypoelastoplastic and hyperelastoplastic problems. Two formulations based on the co-rotational and multiplicative decomposition-based frameworks are implemented for hypoelastoplasticity and hyperelastoplasticity, respectively. Two new terminologies, the so-called standard surface geometry stiffness and the penalty surface geometry stiffness, are introduced and derived through consistently linearizing the virtual work contributed from interior surface integrals. The performance of our DG formulation has been demonstrated through solving a cantilever beam problem undergoing large rotations, as well as a bipolar void coalescence problem where the voids grow up to several hundred times of their original volumes. Fast convergence rates for Newton's iterations have been achieved in our IIPG implementation.

1. Introduction

It is the objective of this paper to establish a consistent spatial formulation for discontinuous Galerkin (DG) methods applied to solid mechanics problems with finite deformation. Several good DG features such as locking-free for nearly incompressible materials suggest a great potential for DG methods to be used as an alternative to CG methods. On the other hand, in general, a full DG discretization for the entire domain may be expensive. For many practical applications, the coupled use of CG and DG methods has CPU advantages through locally employing DG elements. As shown in [19,45], DG methods provide a natural computational framework for modeling crack opening and shear band problems. Obviously, the use of DG elements only in areas near cracks or shear bands is much more efficient. In such situations, DG methods should be formulated in a framework that is consistent with CG methods. Furthermore, it is important to do this in both co-rotational and multiplicative decomposition-based frameworks for solving nonlinear solid mechanics problems with finite deformation. In this paper we develop a consistent spatial formulation for the IIPG method. Rather than demonstrating some specific advantages of DG over CG, numerical examples in this paper are selected to evaluate the performance of the IIPG method for solving large rotation and large deformation problems in terms of accuracy and convergence of the Newton's iteration.

The foundation for modeling finite deformation problems is the theory of nonlinear continuum mechanics [53,17,32,42]. The dominant finite element frameworks are CG-based. The pioneering CG nonlinear finite element analysis for nonlinear solid and structure continua has been developed by Oden in [40]. The work by Hughes and Pister [24] put forward the consistent linearization concept for substantially accelerating solutions for nonlinear problems in CG frameworks. For both hypoelastoplasticity and hyperelastoplasticity, the stress updating schemes and consistent algebraic modulus systematically obtained from the so-called local material integrators have been developed by [25,36,46,43,47,1,48,49,35] for classic

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J_2 and pressure-sensitive plasticity models. The importance of these consistently formulated material integrators is in that complex practical elastoplasticity problems could be solved in tens or hundreds of loading steps rather than tens of thousands of loading steps required by the methods using continuous tangent elastoplastic modulus. This has motivated many commercial finite element codes to develop and implement consistently formulated material integrators.

Parallel to the development of CG methods, DG methods [38,14,4,55,2] have been proposed for reducing the errors induced by the strong implementation of Dirichlet boundary conditions in CG methods. A number of DG formulations for fluid problems have been presented in [9,41,7,10]. For elliptic problems, Arnold et al. [3] proposed a unified theoretical DG framework. For linear elasticity with small deformation, Hansbo and Larson [18], Riviere and Wheeler [44], Wihler [56], Lew et al. [26], and Liu et al. [28] demonstrated that DG methods are good alternatives to CG methods in avoiding volume locking issues. DG methods have been extended to nonlinear problems in the framework of small deformation by Wells et al. [54,34] for damage mechanics problems, by Liu et al. [27,29,30] for poromechanics problems, and by Hansbo [19], Liu et al. [31], and Djoko et al. [12,13] for classic plasticity problems with small deformation. For nonlinear solid mechanics with finite deformation, DG methods have been studied by Noels et al. [39] and Ten Eyck et al. [50,51] for hyperelasticity and by McBride et al. [33] for finite gradient plasticity problems.

We now address the contributions of this work. First, the total Lagrange formulation and the updated Lagrange (spatial) formulation are the two finite element frameworks for solving finite deformation problems. These two methods are theoretically equivalent. For practical applications, however, the spatial formulation is more popular in many commercial finite element codes. This is because the field variables obtained from the total Lagrange formulation are based on the reference configuration and have to be transformed into variables defined in the current configuration for visualization and data analysis purpose. On the other hand, the field variables, i.e., the Cauchy stress and the true stress, are naturally computed in the current configuration and can be directly visualized without any transformation. We therefore adopt a spatial DG formulation. This approach would also greatly facilitate the coupling between CG and DG methods. Detailed DG spatial formulations and implementations for solid mechanics problems with finite deformation are little documented in the literature. In this work, our DG formulation and linearization are performed on the current configuration through employing the spatial velocity and the material time derivative techniques.

Second, the co-rotational formulation [40] for hypoelastoplastic models and potential energy function-based formulation for hyperelastoplastic models are two major finite element frameworks for finite deformation problems. For practical applications, these two frameworks are equally important. The DG methods in [39,50,51,33] are formulated and tested on only hyperelasticity or hyperelastoplasticity. The robust implementation of the co-rotational formulation for finite hypoelasticity has been one of the most challenging topics for computer programming [21,22,52]. In this work, our DG methods are formulated and evaluated on both co-rotational and multiplicative decomposition-based frameworks, which are critical in many practical applications. Third, our DG spatial formulation for finite deformation problems is based on consistently linearizing nonlinear equations, which provides a fast convergence rate for Newton's iterations. In [39], the proposed DG method is symmetric and only for hyperelasticity. As discussed in [31], a family of DG methods, except for the Incomplete Interior Penalty Galerkin [11], have difficulties in achieving a consistent formulation for plasticity problems even with small deformation. A consistent DG formulation derived in [31] for classical plasticity problems is based on IIPG method, but only for problems with small deformation. In this work, we extend IIPG method to finite elastoplastic problems. Finally, the stiffness contributed from pressure loadings is also considered in our DG formulation, which is important for achieving fast convergence rates for Newton's iterations for modeling pressure vessel problems.

We organize the remaining sections of this paper as follows. In Section 2, we summarize the fundamentals of nonlinear continuum mechanics. The material time derivatives of a few deformation-related variables are summarized in this section for facilitating the linearization of our DG formulation. Mathematical statements for modeling finite deformation problems are defined in Section 3. We develop the spatial IIPG formulation in Section 4. The IIPG nonlinear equations are linearized in Section 5. Section 6 addresses the importance of establishing local material integrators for achieving consistent DG formulations. The spatial DG implementation and nonlinear solution procedures are discussed in Section 7. In Section 8, we present numerical examples to demonstrate the performance of our proposed IIPG method. Conclusions are summarized in Section 9.

2. Fundamentals of nonlinear continuum mechanics

In this section, referring to [32,49], we describe some key variables in nonlinear continuum mechanics. These include the deformation gradient, polar decomposition of the deformation gradient, and pairs of stress and strain measurements. More specifically, we summarize the rate change forms of the deformation gradient, infinitesimal volume, and infinitesimal surface area, which will greatly facilitate our linearization of the virtual work in DG frameworks performed in later sections. Objective stress rates are also summarized in this section.

2.1. Strain and stress measurements

Let $B_0 \subset R^3$ be the reference configuration and let $B_t \subset R^3$ be the current deformed configuration. As shown in Fig. 1, a one-to-one mapping $\phi(X,t)$ maps a particle $X \in B_0$ into $x \in B_t$:

$$x = \phi(X,t). \tag{1}$$

The material velocity V and spatial velocity v of the motion ϕ are defined as follows:

$$V(X,t) = \frac{\partial \phi(X,t)}{\partial t}, \quad v(x,t) = V(X,t) \circ \phi^{-1}(X,t), \tag{2}$$

where ϕ^{-1} is the inverse of the mapping function ϕ. The deformation gradient F is defined as the partial derivative of the mapping function ϕ with respective to the reference coordinates as follows:

$$F(X,t) = \frac{\partial \phi(X,t)}{\partial X}.$$

The deformation gradient F can be multiplicatively decomposed into a rotational tensor $R(X,t)$ and a stretch tensor $U(X,t)$ as follows:

$$F(X,t) = R(X,t)U(X,t), \tag{3}$$

where $U(X,t)$ is the right stretch tensor. The above equation is the polar decomposition of the deformation gradient and the rotational tensor $R(X,t)$ plays a key role in establishing co-rotational finite element frameworks for hypoelastoplasticity. The right Cauchy–Green tensor C and left Cauchy–Green tensor b are defined in terms of the deformation gradient F as follows:

$$C(X,t) = F^T F; \quad b(x,t) = FF^T,$$

where the superscript T indicates the transpose operation of tensors. Besides of the right Cauchy–Green and left Cauchy–Green