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# Multigrid methods for isogeometric discretization

K.P.S. Gahalaut, J.K. Kraus, S.K. Tomar *

*Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenbergerstrasse 69, 4040 Linz, Austria*

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ABSTRACT

We present (geometric) multigrid methods for isogeometric discretization of scalar second order elliptic problems. The smoothing property of the relaxation method, and the approximation property of the intergrid transfer operators are analyzed. These properties, when used in the framework of classical multigrid theory, imply uniform convergence of two-grid and multigrid methods. Supporting numerical results are provided for the smoothing property, the approximation property, convergence factor and iterations count for *V*-, *W*- and *F*-cycles, and the linear dependence of *V*-cycle convergence on the smoothing steps. For two dimensions, numerical results include the problems with variable coefficients, simple multi-patch geometry, a quarter annulus, and the dependence of convergence behavior on refinement levels $\ell$, whereas for three dimensions, only the constant coefficient problem in a unit cube is considered. The numerical results are complete up to polynomial order $p = 4$, and for $C^0$ and $C^{p-1}$ smoothness.



## 1. Introduction

Isogeometric method (IGM), introduced in 2005 [28], aims to bridge the gap between finite element method (FEM) and computer aided design (CAD). The main idea of IGM is to directly use the geometry provided by the CAD system, and following the isoparametric approach, to approximate the unknown variables of differential equation by the same functions which are used in the CAD system. IGM offers several advantages when compared to classical FEM. For example, some common geometries arising in engineering and applied sciences, such as circles or ellipses, are represented exactly, and complicated geometries are represented more accurately than traditional polynomial based approaches. Another noteworthy advantage of IGM over classical FEM is the higher continuity. It is a difficult and cumbersome (if not impossible) task to achieve even $C^1$ inter-element continuity in FEM, whereas IGM offers up to $C^{p-m}$ continuity, where $p$ denotes the polynomial order and $m$ denotes the knot-multiplicity.

A primary goal of IGM is to be geometrically precise at the coarsest discretization level. In particular, the description of the geometry, taken directly from the CAD system, is incorporated exactly at the coarsest mesh level. This eliminates the necessity of further communication with the CAD system when mesh refinement is carried out. Thereby, the mesh refinement does not modify the geometry. There are several computational geometry technologies that could serve as a basis for IGM. However, non-uniform rational B-splines (NURBS) are the most widely used and well established computational technology in CAD, which we shall also pursue in this work. In last several years IGM has been applied to a variety of problems, e.g., fluid dynamics, electromagnetics, structural mechanics, etc. with promising results. For a detailed discussion see early papers on IGM [2,8–11,18,19] and the book [17]. Since the introduction, most of the IGM progress has been focused on the applications and discretization properties. Nevertheless, when dealing with large problems, the cost of solving the linear system of equations arising from the isogeometric discretization becomes an important issue. Clearly, the discretization matrix $A$ gets denser with increasing $p$. Therefore, the cost of a direct solver, particularly for large problems, becomes prohibitively expensive. This necessitates the development and use of fast and efficient iterative solvers. It is known that the performance of iterative solvers depends on the condition number of the matrix $A$. Let $\kappa = \lambda_{\max}/\lambda_{\min}$ (i.e. ratio of largest to smallest eigenvalues) denote the spectral condition number of $A$. In Table 1, we present $\kappa(A)$ of the Laplace operator. We consider a unit square domain and a uniform mesh of $n_0 \times n_0$ elements (open knot-spans for IGM) with mesh-size $h$. This also serves as a comparison between FEM with Lagrange basis[1] and IGM. For a fair comparison, we take $C^0$ continuity in IGM as this results in the same problem size for both the methods. Though the condition number for both the methods reaches $\mathcal{O}(h^{-2})$ asymptotically, however, for IGM the polynomial order $p$ clearly affects the range of the mesh when asymptotic behavior is reached. For example, for IGM with $p = 5$, the asymptotic behavior is not

* Corresponding author. Tel.: +43 732 24685220 (Off.); fax: +43 732 24685212.
*E-mail addresses:* krishan.gahalaut@ricam.oeaw.ac.at (K.P.S. Gahalaut), johannes.kraus@ricam.oeaw.ac.at (J.K Kraus), satyendra.tomar@ricam.oeaw.ac.at (S.K. Tomar).

[1] Alternatively, the hierarchical basis [37] can also be used for very good condition numbers, but the inter-element continuity is still $C^0$.

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**Table 1**
Comparison of $\kappa(A)$.

| $n_0$ | $p=2$ | | $p=5$ | |
|---|---|---|---|---|
| | FEM | IGM | FEM | IGM |
| 2 | 14 | 7 | 581 | 11094 |
| 4 | 55 | 12 | 2317 | 12951 |
| 8 | 216 | 36 | 9263 | 13680 |
| 16 | 859 | 140 | 37050 | 13886 |
| 32 | 3434 | 554 | 148198 | 13939 |
| 64 | 13734 | 2215 | 592789 | 13952 |

reached up to a reasonably refined mesh. On one hand, this is an advantage as the condition numbers are moderate towards the finer spectrum of the mesh, but on the other hand, this is a serious disadvantage towards the coarser spectrum of the mesh. Note that the condition number rapidly increases with $p$, and it can reach $\sim 10^9$ for $p = 10$ even for $n_0 = 2$. This is also reflected by the bound of $\kappa$ which behaves like $\mathcal{O}(p^{2d}4^{dp})$, see [24], where $d$ denotes the dimension of the problem domain.

To the best of authors' knowledge, so far there are only very few papers [12,13,16,29] which address the performance of linear algebra solvers. In Ref. [16], the authors study the performance of direct solvers which are clearly not suitable for large problems, specially in three-dimensions. In Ref. [29], the tearing and interconnecting approach of finite element methods is used in the context of isogeometric analysis, and the numerical tests (in absence of any theoretical study) suggest almost optimal (with a logarithmic factor) convergence rates of the proposed isogeometric tearing and interconnecting method. The only paper which provides rigorous theoretical study, supported by extensive numerical examples, is by Beirao et al. [12] where the authors discuss the overlapping Schwarz methods. The same authors have also proposed BDDC preconditioners for isogeometric analysis in [13].

In this paper we address another class of linear algebra solvers with optimal complexity, namely multigrid methods. During the last five decades (first paper by Fedorenko in 1961), these methods have been established as a powerful and efficient tool for solving linear system of equations arising in a variety of problems [5,26,38]. The key idea of multigrid goes back to R.P. Fedorenko in the early 60s [22,23], who developed the first multigrid method for solving the Poisson equation on a unit square. The first rigorous convergence proof was provided by Bakhwalov [4]. In early 70s, the multigrid idea was generalized to variational finite difference equations and general finite element equations by Astrachancev [1] and Korneev [30]. However, the huge potential of multigrid methods was realized due to the works of Brandt [6] and Hackbusch [25,26]. A few years later, in the early eighties, algebraic multigrid methods were introduced by Brandt et al. [7], which rebuild the multigrid algorithm based on the information that is accessible via the system of (linear) algebraic equations only. For a more recent exposition of multigrid methods in a multilevel block factorization framework, see also [39].

Our focus in this paper is on multigrid methods for solving the linear system of equations arising from the isogeometric discretization of scalar second order elliptic problems in a single patch. We first prove the condition number estimates of the discrete system for the $h$-refinement, and provide the supporting numerical results for all levels of smoothness (from $C^0$ to $C^{p-1}$). These results suggest the expected behavior from the two-(multi-) grid solver. We then prove both the components of the two-grid solver, namely the *approximation property* of the inter-grid transfer operators, and the *smoothing property* of the classical Gauss–Seidel (symmetric as well as non-symmetric) method. Together, these two components establish the $h$-independence of the two-grid solver. For the multi-grid solver, which uses the two-grid solver recursively, we recall the $h$-independent convergence estimates from [26].

Following the terminology of traditional FEM, we will call the open knot-span as element wherever appropriate. Moreover, as most of the NURBS based designs in engineering use polynomial order $p = 2$ and 3, throughout this article we will confine ourselves up to $p = 4$. Furthermore, throughout this article we use the notation $f \preceq g$ (respectively $f \succeq g$) to denote $f \leqslant cg$ (respectively $f \geqslant cg$) where the constant $c$ is independent of the mesh parameter $h$ and the inequality arguments, but it may depend on the polynomial order $p$.

The contents of this article are organized as follows. In Section 2 we briefly recall the notations for B-splines and NURBS. The geometry mapping and the function spaces are also introduced there. In Section 3 we describe the model problem and recall error estimates. Furthermore, the properties of the discrete system and the norm equivalences are also studied there. In Section 4 we discuss the two-grid method. The multigrid method is then discussed in Section 5. Numerical results on four model problems are presented in Section 6. Finally, some conclusions are drawn in Section 7.

## 2. Notations

To keep the article self-contained, we briefly recall the definitions of B-splines and NURBS. For the properties of B-splines and NURBS, which are related to our problem, the reader is referred to [17,28]. For a detailed exposition see, e.g., [32,34,36]. Let $p$ be a non-negative integer denoting the polynomial order, and $n$ be the number of basis functions (B-splines or NURBS). With $i = 1, 2, \ldots, n + p + 1$, denoting the knot index, we assume that the knot vector $\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$ is a sequence of non-decreasing knots $\xi_i$. The knot vector is uniform if the knots are equally spaced, and it is non-uniform when the knots are unequally placed. It is also possible for more than one knot to have the same value, wherein they are called multiple knots. A knot vector is said to be open if its first and last knot values appear $p + 1$ times.

The B-spline basis functions, denoted by $B_i^p(\xi)$, are defined recursively as follows:

$$B_i^0(\xi) = \begin{cases} 1 & \text{if } \xi_i \leqslant \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}, \tag{1a}$$

$$B_i^p(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} B_i^{p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} B_{i+1}^{p-1}(\xi). \tag{1b}$$

Note that for non-repeated internal knots the support of a B-spline basis function of order $p$ is always $p + 1$ knot spans, and every knot span is shared by $p + 1$ B-spline basis functions, see Fig. 1 where we plot B-spline basis functions for open, uniform knot vector $\{0, 0, \cdots, \frac{1}{16}, \frac{1}{8}, \cdots, \frac{7}{8}, \frac{15}{16}, \cdots, 1, 1\}$ with order 2 and 8. The basis functions formed from open knot vectors are interpolatory at the ends of the parameter space interval $[\xi_1, \xi_{n+p+1}]$. In general, basis functions of order $p$ have $p - m_i$ continuous derivatives across knot $\xi_i$, where $m_i$ is the multiplicity of the value $\xi_i$ in the knot vector. When the multiplicity of an internal knot value is exactly $p$, the basis is interpolatory at that knot. This is an important property of B-spline basis functions, in particular, from analysis point of view. Moreover, in IGM the geometry is fixed at the coarsest level of discretization, and any subsequent refinement (whether $h-$, $p-$ or $r-$) does not change it. For example, if a partition $\mathcal{Q}_{h_0}$ of $(0, 1)$ is given with the knot vector $\Xi_0 = \{0, 0, 0, 0, 1/2, 1, 1, 1, 1\}$, then the refined partition $\mathcal{Q}_{h_1}$ can be obtained from $\mathcal{Q}_{h_0}$ via a regular subdivision of knot vector $\Xi_0$ into $\Xi_1$, where $\Xi_1 = \{0, 0, 0, 0, 1/4, 1/2, 3/4, 1, 1, 1, 1\}$. Further refinements are similarly carried out.