



From face to element unknowns by local static condensation with application to nonconforming finite elements[☆]

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ABSTRACT

We derive in this paper a new local static condensation strategy which allows to reduce significantly the number of unknowns in algebraic systems arising in discretization of partial differential equations. We apply it to the discretization of a model linear elliptic diffusion and a model nonlinear parabolic advection–diffusion–reaction problem by Crouzeix–Raviart nonconforming finite elements. Herein, the unknowns, originally associated with the mesh faces, can be reduced to new unknowns associated with the mesh elements. The resulting matrices are sparse, with possibly only four nonzero entries per row in two space dimensions, positive definite in dependence on the mesh geometry and the diffusion–dispersion tensor, but in general nonsymmetric. Our approach consists in introducing new element unknowns, the identification of suitable local vertex-based subproblems, and the inversion of the corresponding local matrices. We give sufficient conditions for the well-posedness of the local problems, as well as for the resulting global one. In addition, we provide a geometrical interpretation which suggests how to influence the form of the local and global matrices depending on the local mesh and data. We finally present an abstract generalization allowing for a further reduction of the number of unknowns, typically to one unknown per a set of mesh elements. We conclude by numerical experiments which show that the condition number of the resulting matrices is quite insensitive to the mesh anisotropies and the diffusion tensor inhomogeneities.

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1. Introduction

Let $\mathbb{Z}^i$ and $\mathbb{Z}^b$ be two given matrices and let E be a given right-hand side vector. We consider in this paper the following problem: find $\Lambda, \Lambda := (\Lambda^i, \Lambda^b)^t$, such that

$$\begin{pmatrix} \mathbb{Z}^i & \mathbb{Z}^b \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} \Lambda^i \\ \Lambda^b \end{pmatrix} = \begin{pmatrix} E \\ 0 \end{pmatrix}. \quad (1.1)$$

Here $\mathbb{I}$ stands for the identity matrix. Note that it follows from (1.1) that $\Lambda^b = 0$, so that (1.1) can be equivalently rewritten as: find Λ^i such that

$$\mathbb{Z}^i \Lambda^i = E. \quad (1.2)$$

System (1.1) typically results in the discretization of elliptic or parabolic problems by the Crouzeix–Raviart nonconforming finite element method of Crouzeix and Raviart [6] or by the mixed finite element method of Raviart and Thomas [10] and [3], see Sections 3 and 4 for details. Therein, $\mathbb{Z}^i$ is a sparse (symmetric) positive definite matrix.

The purpose of this paper is to devise a general principle allowing to *reduce equivalently* system (1.1) to a system

$$\mathbb{S}P = H, \quad (1.3)$$

with a sparse and easily computable matrix $\mathbb{S}$ and *much fewer unknowns* P . Let Ω be a computational domain and $\mathcal{T}_h$ its simplicial mesh. We suppose that the unknowns Λ are associated with the mesh faces, whereas the new unknowns P are associated with the mesh elements.

In Section 2, we introduce an *abstract* and *algebraic* principle for reducing (1.1) to (1.3). We introduce an *arbitrary matrix* $\mathbb{N}$ and augment (1.1) with the relation $\mathbb{N}\Lambda = P$. We then identify suitable local subproblems of the augmented system, invert the corresponding local matrices, obtain local expressions of the unknowns Λ^i in terms of the new unknowns P , and finally identify the matrix $\mathbb{S}$ and the right-hand side vector H of (1.3). Under Assumption 2.2

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below, we also prove the well-posedness of (1.3), the equivalence of (1.3) with (1.1), and characterize the stencil (maximal number of nonzero entries per each matrix row) of $\mathbb{S}$. This approach generalizes those obtained in the framework of the mixed finite element method in Younès et al. [19,18], Chavent et al. [4], and [14,17]. In contrast to [19,18], our starting point is purely algebraic, and no geometrical mesh or discretization information is used. Moreover, different choices of $\mathbb{N}$ are possible and a family of reformulations can be obtained.

Enlightening the abstract algebraic approach of Section 2, we provide in Section 3 its specification for the discretization of a model linear elliptic diffusion problem by the Crouzeix–Raviart nonconforming finite element method. We also present its geometric interpretation. Here, the new element values are the values of the Crouzeix–Raviart approximation in points associated with the elements (not necessarily inside the elements). We give in Section 3 sufficient conditions in terms of the diffusion tensor $\mathbf{S}$ and of the geometry of the mesh $\mathcal{T}_h$ for Assumption 2.2 to hold. We also investigate how the properties of the local problems and of the matrix $\mathbb{S}$ can be influenced by the choice of the element points. In Section 4, we then present similar developments for a model nonlinear parabolic advection–diffusion–reaction problem discretized by the Crouzeix–Raviart method.

In Section 5, we report results of several numerical experiments. The matrices $\mathbb{S}$ resulting from our approach are in general positive definite but nonsymmetric. Their condition number is in our numerical examples quite insensitive to the anisotropies of the mesh $\mathcal{T}_h$ and inhomogeneities of the tensor $\mathbf{S}$ for linear diffusion problems. We observe CPU gains for both standard direct and iterative solvers in the range of a factor 1.5 to 3, 30 in particular situations. A concluding discussion is given in Section 6.

We finish the paper by Appendix A which gives a generalization of the approach of Section 2, weakens Assumption 2.2, and enables a further reduction of the number of unknowns.

2. Static condensation from edges to elements

We introduce in this section our basic static condensation principle, which enables us to rewrite (1.1) equivalently as (1.3), reducing the number of unknowns from mesh faces to mesh elements.

2.1. The domain and its mesh

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a polygonal (polyhedral) domain and let $\mathcal{T}_h$ be a matching (containing no hanging nodes) simplicial mesh of Ω in the sense of Ciarlet [5]. We denote by $\mathcal{E}_h$ the set of all $(d-1)$ -dimensional faces of $\mathcal{T}_h$. We divide $\mathcal{E}_h$ into interior faces $\mathcal{E}_h^i$ and boundary faces $\mathcal{E}_h^b$. For $\sigma \in \mathcal{E}_h$, let $\mathbf{x}_\sigma$ stand for the barycenter of the face σ . We next denote by $\mathcal{V}_h$ the set of vertices of $\mathcal{T}_h$. For a given vertex $V \in \mathcal{V}_h$, we shall denote by $\mathcal{T}_V$ the patch of the elements of $\mathcal{T}_h$ which share V , by $\mathcal{E}_V^{\text{int}} \subset \mathcal{E}_h^i$ the interior faces of $\mathcal{T}_V$, and by $\mathcal{E}_V^{\text{ext}}$ the faces of $\mathcal{T}_V$ not having V as vertex. We set $\mathcal{E}_V := \mathcal{E}_V^{\text{int}} \cup \mathcal{E}_V^{\text{ext}}$ and we also denote by $\mathcal{E}_V^b$ the faces of $\mathcal{T}_V$ which lie on the boundary $\partial\Omega$ and not in $\mathcal{E}_V$. We refer to Figs. 1–3 for an illustration in two space dimensions. Let $K \in \mathcal{T}_h$. By $\mathcal{E}_K$, we denote the set of all faces of K and by $\mathcal{E}_K^i$ the set of such faces of $\mathcal{E}_K$ which lie in $\mathcal{E}_h^i$. Let V be a vertex of K . We will also employ the notation $\mathcal{E}_{V,K}$ for the faces of K which have V as vertex. Finally, $\mathbf{x}_K$ denotes the barycenter of K . The symbol $|S|$ stands for the cardinality (the number of elements) of a set S .

2.2. Augmented problem setting

Let $\mathbb{Z}^i \in \mathbb{R}^{|\mathcal{E}_h^i| \times |\mathcal{E}_h^i|}$, $\mathbb{Z}^b \in \mathbb{R}^{|\mathcal{E}_h^i| \times |\mathcal{E}_h^b|}$ be given matrices. Both have the number of rows equal to the number of mesh interior faces; the number of columns of $\mathbb{Z}^i$ is given by the number of mesh interior

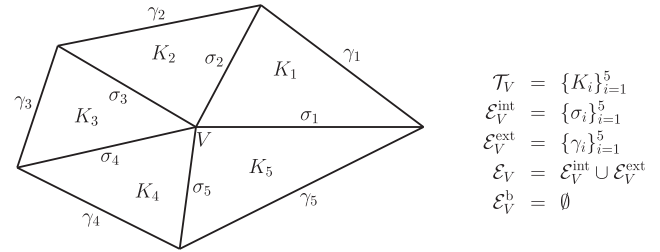


Fig. 1. An example of a patch $\mathcal{T}_V$ around a vertex V in the interior of Ω .

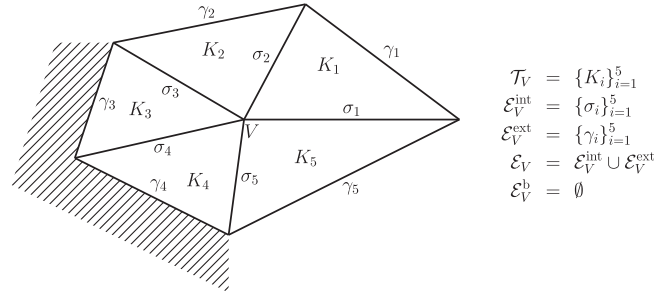


Fig. 2. An example of a patch $\mathcal{T}_V$ around a vertex V close to the boundary of Ω .

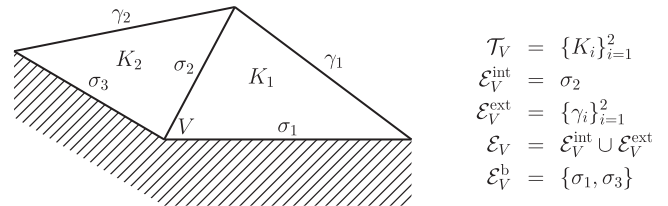


Fig. 3. An example of a patch $\mathcal{T}_V$ around a vertex V on the boundary of Ω .

faces, whereas that of $\mathbb{Z}^b$ by that of mesh boundary faces. Let a right-hand side vector $E \in \mathbb{R}^{|\mathcal{E}_h^i|}$ be also given. We consider the following problem: find $\Lambda \in \mathbb{R}^{|\mathcal{E}_h|}$, $\Lambda = \{\Lambda_\sigma\}_{\sigma \in \mathcal{E}_h} = (\Lambda^i, \Lambda^b)^t$, such that (1.1) holds. Herein, we merely suppose that (1.1) is well posed, i.e., that the system matrix of (1.1) is nonsingular, and that on a row associated with a face $\sigma \in \mathcal{E}_h^i$, the only nonzero entries of $(\mathbb{Z}^i, \mathbb{Z}^b)$ lie on columns associated with faces $\gamma \in \mathcal{E}_h$ such that σ and γ belong to the same simplex.

Let $\mathbb{N} \in \mathbb{R}^{|\mathcal{T}_h| \times |\mathcal{E}_h|}$. We suppose that on a row of $\mathbb{N}$ associated with an element $K \in \mathcal{T}_h$, the only nonzero entries are on columns associated with the faces $\mathcal{E}_K$ of K ; apart from this assumption, the matrix $\mathbb{N}$ is arbitrary. This assumption will ensure locality of our approach and sparsity of the final matrix $\mathbb{S}$. Introduce one new unknown P_K for each mesh element $K \in \mathcal{T}_h$. Let P be the corresponding algebraic vector, $P = \{P_K\}_{K \in \mathcal{T}_h}$. Consider now the following augmented problem: find $\Lambda \in \mathbb{R}^{|\mathcal{E}_h|}$ and $P \in \mathbb{R}^{|\mathcal{T}_h|}$ such that (1.1) holds together with

$$\mathbb{N}\Lambda = P. \quad (2.1)$$

This problem is well posed in the sense that there exists a unique solution (Λ, P) of (1.1), (2.1). Indeed, (1.1) defines Λ in a unique way thanks to its well-posedness. As (2.1) is completely uncoupled from (1.1), P is simply prescribed by $P := \mathbb{N}\Lambda$.

2.3. Structure of the algorithm

In order to present the key idea of our approach as clearly as possible and to underline its simplicity, we now present the structure of our algorithm:

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