Comput. Methods Appl. Mech. Engrg. 247–248 (2012) 65–72

Contents lists available at SciVerse ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

On stabilized finite element methods based on the Scott–Zhang projector. Circumventing the inf–sup condition for the Stokes problem

Santiago Badia *

Centre Internacional de Mètodes Numèrics en Enginyeria (CIMNE), Parc Mediterrani de la Tecnologia, Esteve Terrades 5, 08860 Castelldefels, Spain
Universitat Politecnica de Catalunya, Jordi Girona 1-3, Edifici C1, 08034 Barcelona, Spain

ARTICLE INFO

Article history:
Received 14 May 2012
Received in revised form 16 July 2012
Accepted 31 July 2012
Available online 16 August 2012

Keywords:
Stabilized finite elements
Stokes problem
Indefinite systems

ABSTRACT

In this work we propose a stabilized finite element method that permits us to circumvent discrete inf–sup conditions, e.g. allowing equal order interpolation. The type of method we propose belongs to the family of symmetric stabilization techniques, which are based on the introduction of additional terms that penalize the difference between some quantities, i.e. the pressure gradient in the Stokes problem, and their finite element projections. The key feature of the formulation we propose is the definition of the projection to be used, a non-standard Scott–Zhang projector that is well-defined for $L^1(\Omega)$ functions. The resulting method has some appealing features: the projector is local and nested meshes or enriched spaces are not required.

1. Introduction

Many physical problems in science and engineering are modeled with partial differential equations without a coercivity property, e.g. the (Navier–)Stokes equations for incompressible flows, Darcy's problem for flux in porous media, and some versions of the Maxwell equations. At the continuous level, these saddle-point problems are well posed by virtue of some inf–sup condition. As model problem, let us consider Stokes' system on a bounded open domain Ω with homogeneous boundary conditions:

$$-\nu\Delta u + \nabla p = f, \quad \nabla\cdot u = 0, \quad u|_{\partial\Omega} = 0, \tag{1}$$

where u is the velocity, p the pressure, ν the fluid viscosity and f the body force. Using standard notation, we can state the problem in weak form: find $(u,p) \in H_0^1(\Omega)\times L_0^2(\Omega)$ such that

$$c(u,p,v,q) := \nu(\nabla u,\nabla v) - (p,\nabla\cdot v) + (q,\nabla\cdot u) = \langle f, v\rangle =: \ell(v) \tag{2}$$

for any $(v,q)\in H_0^1(\Omega)\times L_0^2(\Omega)$. For the Stokes problem, pressure stability relies on the following inf–sup condition: for any $p\in L_0^2(\Omega)$, there exists a $v_p \in H_0^1(\Omega)$ with unit norm such that $\beta\|p\| \geqslant (p,\nabla\cdot v_p)$, for some $\beta>0$. Unlike coercivity, inf–sup conditions are not inherited by sub-spaces of functions, complicating Galerkin approximations. We have to explicitly build finite element (FE) spaces that satisfy discrete versions of the inf–sup conditions, and appealing choices such as equal-order interpolation for all the unknowns cannot be used [8]. The situation is slightly more involved in general, since the well-posedness of saddle-point problems only requires coercivity on the kernel of the constraint operator, e.g. the Maxwell and Darcy problems [14].

The use of inf–sup stable formulations can be particularly impractical in multi-physics simulations of saddle-point systems, i.e. inductionless magnetohydrodynamics (MHD) (coupling Stokes and Darcy-type problems, see [21]) or incompressible visco-resistive MHD (coupling Stokes and Maxwell-type problems, see [3]). This approach requires different FE spaces for the different unknowns [22], complicating the integration subroutines and the matrix graph (a different graph for every block of the full matrix is needed). In other situations, in which different asymptotic limits of the problem (in terms of the physical parameters) lead to different saddle-point systems, e.g. the Brinkman model for creeping flow in porous media, the inf–sup stable approximation cannot lead in general to unconditional stability, since different limits require different FE spaces (see [1]).

Alternatively, we can consider stabilized FE techniques, which consist in the introduction of additional (stabilization) terms that provide the numerical method with the proper stability without the need to satisfy discrete inf–sup conditions. Initially, the stabilization terms were based on the residual at FE interiors, as in the popular Galerkin/least-squares (GLS) method [16] and the improved Variational Multiscale (VMS) method [17] proposed by Hughes and co-workers. Let us consider a partition $\mathcal{T}_h$ of Ω into tetrahedra/hexahedra, denoted by K, and conforming FE spaces $V_h\times Q_h \subset H_0^1(\Omega)\times L_0^2(\Omega)$. The stabilized methods GLS ($\theta=-1$) and VMS ($\theta=1$) read as:

* Address: Universitat Politecnica de Catalunya, Jordi Girona 1-3, Edifici C1, 08034 Barcelona, Spain.
E-mail address: sbadia@cimne.upc.edu

0045-7825/$ - see front matter © 2012 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.cma.2012.07.020

$$c(u_h, v_h, p_h, q_h) + \sum_{K\in\mathcal{T}_h} \delta_K(-\Delta u_h + \nabla p_h, \theta\Delta v_h + \nabla q_h)_K = \ell(v_h) + \sum_{K\in\mathcal{T}_h} \delta_K\langle f, \theta\Delta v_h + \nabla q_h\rangle_K, \quad (3)$$

where δ_K is a numerical parameter to be defined later. These methods allow equal-order interpolation, are consistent and exhibit optimal convergence rates. On the other side, they are usually criticized for giving unphysical pressure boundary layers [5], for the additional cost involved in the evaluation of higher order derivatives and the weak inconsistency for first order approximations [18], the fact that the forcing term is also affected by the stabilization and the hard extension to transient problems [13], usually carried out via expensive space–time FEs (see e.g. [24]). Probably, the main shortcoming of GLS, VMS and related residual-based formulations is manifested when dealing with multi-physics applications. These stabilized FE formulations include a large number of additional coupling terms, which fill blocks that are zero for the Galerkin method; see [21,3] for MHD applications. Another important problem related to these methods is the fact that they destroy the skew-symmetric form of the off-diagonal blocks (stabilized gradient and divergence matrix) in the Navier–Stokes and MHD systems, making pressure-segregation (fractional step) methods to lose their unconditional stability (see [4]).

The introduction of symmetric stabilization techniques represented one step further in the improvement of FE stabilization techniques, since they solve all the problems commented above. Instead of considering residual-based terms, these methods introduce penalty terms over the difference between some quantities, i.e. the pressure gradient for the Stokes problem, and their projections. This family of methods does not perturb the right-hand side of the problem, and stabilizes the bilinear form as follows:

$$c(u_h, v_h, p_h, q_h) + \sum_{K\in\mathcal{T}_h} \delta_K(\nabla p_h - \pi_h(\nabla p_h), \nabla q_h - \pi_h(\nabla q_h))_K = \ell(v_h),$$

where $\pi_h(\cdot)$ is a FE projector; different definitions for $\pi_h(\cdot)$ lead to different techniques. The resulting method is only weakly consistent, i.e. the stabilization term does not cancel for the exact solution but vanishes as the mesh size $h \searrow 0$ in such a way that optimal convergence is kept. Motivated by the inherited stability of fractional step methods, Codina and Blasco provided in their pioneering work [12] the first algorithm of this kind, based on the $L^2(\Omega)$-orthogonal projector, coined orthogonal subscales (OSS). Unfortunately, this projector is global, i.e. the stabilization term leads to a dense matrix. Certainly, the method is never computed this way, and the projection is usually sent to the right-hand side of the linear system. In case of solving transient problems, it can simply be treated explicitly. In those situations, for reasonably small time step sizes, the OSS method has perfect sense and it is an effective and simple algorithm, since the CPU cost per time step used for the computation of the global projections in negligible. On the contrary, to send the projection term to the right-hand side, and make it implicit via Richardson iterations (usually merged with nonlinear iterations [11]) is not advisable unless a very small time-step size is used, since it can drastically increase the number of nonlinear iterations or simply diverge; this approach is even harder to justify for linear problems as the Stokes system. An alternative consists in dealing with the exact matrix, explicitly solving the mass matrix systems for the projection evaluations at every iteration of an external Krylov solver. This approach is hard to implement and prevent us to use direct solvers, and by extension substructuring domain decomposition techniques with exact local solvers [25].

Becker and Braack envisaged in [5] an original way to avoid the global projections in [12]. Their method was later called local projection stabilization (LPS). The price to pay is a tighten requirement over the mesh partitions: specific hierarchical meshes were needed, since the method is based on the definition of fine and coarse FE spaces. On the other hand, the projection is not over the original FE space, as in [12], but on a discontinuous space of functions. The original LPS formulation has been lately denoted as two-level LPS, due to the requirement of two nested meshes for the definition of the stabilization terms. A one-level LPS formulation has also been designed [19], in which the fine space is attained with an enrichment of the coarse one via additional functions of bubble type. This way, we can eliminate the stringent mesh requirement but now a particular type of enriched FE spaces must be used.

The development of stabilized FE methods that allows one to circumvent inf–sup conditions has been almost entirely developed for the Stokes problem and the nonlinear Navier–Stokes equations. The extension to problems that only present coercivity in the kernel of the constraint operator is more recent. We refer to [2] for a detailed exposition of VMS and symmetric projection stabilization schemes for Darcy's and Maxwell's problems.

The purpose of this work is to present a new method, based on a particular $L^1(\Omega)$ Scott–Zhang projector that shares all the aforementioned benefits of symmetric stabilization techniques as well as: *only local projections* are required, *no assumption over the mesh partition* (e.g. nested meshes) is needed and *no assumption over the FE spaces* (e.g. equal-order Lagrangian FEs can be used without additional enrichment) is needed.

In Section 2, we introduce some notation, as well as the definition of the method and implementation aspects. Section 3 is devoted to the numerical analysis of the algorithm, both stability and *a priori* error estimates. Some numerical tests are presented in Section 4. Finally, we draw some conclusions in Section 5.

2. Definition of the method

Let us consider the Stokes problem (2) for an open, bounded polyhedral domain Ω in $\mathbb{R}^d$, where $d = 2, 3$ is the space dimension. We will use standard notation for Sobolev spaces (see [7]). In particular, the $L^2(\omega)$ scalar product will be denoted by $(\cdot,\cdot)_\omega$ for some $\omega \subset \Omega$, but the domain subscript is omitted for $\omega \equiv \Omega$ (analogously for the duality pairing $\langle\cdot,\cdot\rangle$). The $L^2(\Omega)$ norm is denoted as $\|\cdot\|$. We will define the velocity and pressure spaces as $V_0 \equiv H_0^1(\Omega)$ and $Q \equiv L_0^2(\Omega)$, endowed with the norms $\|v\|_V := \nu^{\frac{1}{2}}\|\nabla v\|$ and $\|q\|_Q := \nu^{-\frac{1}{2}}\|q\|$, properly scaled with the fluid viscosity ν. $C^0(\bar{\Omega})$ denotes the space of continuous functions. We will omit the d superscript in vector-valued functional spaces.

Let us consider now a partition $\mathcal{T}_h$ of Ω into d-simplices, quadrilaterals ($d = 2$) or hexahedra ($d = 3$) where every $K \in \mathcal{T}_h$ is the image of a reference element $\widehat{K}$ through an affine mapping $F_K : \widehat{K} \to K$ (see [9, Chp. 2]); we can assume that every edge of $\widehat{K}$ has length one. $P_r(\widehat{K})$ is the space of complete polynomials of degree r on $\widehat{K}$. For d-simplicial FE partitions, we define the space of element-wise discontinuous functions

$$D_h := \{v_h : v_h|_K \circ F_K \in P_r(\widehat{K}),\ K \in \mathcal{T}_h\}.$$

The continuous FE spaces are obtained by enforcing continuity, namely $V_h := D_h \cap C^0(\bar{\Omega})$ and $Q_h := D_h \cap C^0(\bar{\Omega}) \cap L_0^2(\Omega)$ for the velocity and pressure respectively. We will also make use of the FE space with null trace $V_{h,0} := V_h \cap H_0^1(\Omega)$. The order of approximation r to be used for velocity and pressure approximations can be different. For quadrilaterals and hexahedra, the spaces are obtained by replacing $P_r(\widehat{K})$ by $Q_r(\widehat{K})$, the space of polynomials with maximum degree r in each reference space coordinate on $\widehat{K}$.

For the FE space V_h we denote by $\mathcal{N}_h$ the set of all interpolation nodes related to $\mathcal{T}_h$ and by $\{\phi^a\}_{a\in\mathcal{N}_h}$ the corresponding nodal basis of V_h. We also denote by $\mathcal{N}_h(K)$ the set of all nodes that belong to a FE K. Continuous FE functions can be written as $v_h = \sum_{a\in\mathcal{N}_h} v_h^a \phi^a(\mathbf{x})$, where v_h^a denotes the nodal value of v_h corresponding to a. Analo-