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Weak boundary conditions for wave propagation problems in confined domains: Formulation and implementation using a variational multiscale method ☆

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ABSTRACT

We propose a new approach to the enforcement of Dirichlet, Neumann, or Robin boundary conditions in finite element computations of wave propagation problems. The key idea is to enforce the boundary conditions weakly as part of the variational formulation. Due to the hyperbolic structure of the problem considered, the variational formulation does not require any penalty parameters, in contrast with what typically happens in elliptic or advection–diffusion (parabolic) problems. This article presents the implementation of the proposed boundary condition framework using a variational multiscale method for the wave equation in mixed form. We conclude with an extensive set of tests to validate the robustness and accuracy of the proposed approach.

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1. Introduction

Weak boundary conditions are enforced using the variational formulation associated with a partial differential equation (PDE) problem, rather than directly incorporating the boundary values in the function spaces used to represent the solution.

Weak boundary conditions can be traced back to the late 1960s. Lions [49] considered the problem of solving elliptic PDEs with very rough Dirichlet boundary data, and proposed a formulation in which the Dirichlet boundary condition is replaced by a Robin condition depending on an artificial penalty. Aubin [1] extended this approach in the framework of finite difference approximations of nonlinear problems. A consistent and optimal penalty formulation for Dirichlet boundary conditions for elliptic problems had been proposed by Nitsche [52]. More recently, the authors in [2,3] utilized weak enforcement of boundary conditions for advection–diffusion problems, the Navier–Stokes equations, and turbulence, and showed improved results in the presence of boundary layers.

The computational data structure of a finite element method is often times simplified by the use of weak boundary conditions, as it is not necessary to directly prescribe the values of the numerical solution's boundary degrees of freedom. The reader can already realize this fact considering *no slip* boundary conditions for the Navier–Stokes equations [2,3]. As will be discussed momentarily, the advantages are even greater in the context of wave propagation problems, which require inviscid, zero normal flux boundary conditions, and in nonlinear systems, in which strong enforcement of boundary conditions may lead to lack of discrete conservation principles.

In this work we propose a new approach to weak boundary conditions in the context of an abstract linear wave equation in mixed form, of great importance in time-domain acoustics problems, often arising in seismic inversion/detection applications. In addition, the mixed form of the wave equation under consideration is also a simplified model for more complex nonlinear equations, such as, among many others, the compressible flow equations of Lagrangian shock hydrodynamics [50,55–61], nonlinear acoustics, shallow water flows, and meteorological flows [14,29,30,38–46,48,51,54,62,63].

In particular, we consider the case of media which only allow for longitudinal waves, also referred to as "bulk" or "*P*" waves. We will not consider non-reflecting boundary conditions, for which considerable literature exists (see [15–28,31,47,53] and references therein).

The simplest possible boundary conditions in the context of longitudinal waves involve enforcing either the pressure or the normal velocity component. Although in engineering mechanics the pressure is considered as the diagonal component of a stress tensor, in longitudinal wave problems pressure boundary conditions are of *Dirichlet* type, while the normal velocity boundary

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conditions are of *Neumann* type. As discussed also in Section 2, this situation is somewhat reversed with respect to waves in solids, in which the full velocity component can be imposed as a Dirichlet boundary condition, while the stress boundary condition is of Neumann type. This is due to the fact that the spatial differential operators in the mixed equations for longitudinal waves have a structure analogous to the Darcy flow operator, while spatial differential operators in the mixed equations for waves in solids have a structure similar to the Stokes flow operator.

Codina analyzed in [9] a stabilized method in which *both* pressure and normal velocity boundary conditions were enforced *strongly*. The authors in [4,11–13,35–37], instead, using finite elements of Raviart–Thomas type, enforced the pressure (Dirichlet) boundary condition *weakly* while the normal velocity (Neumann) boundary condition was enforced *strongly*, using a spatial discretization often employed for Darcy flow problems.

Our approach differs from the ones above in that *both* Dirichlet and Neumann boundary conditions are imposed *weakly*. The advantage of this strategy is that equal-order approximations can be employed for pressure and velocity (thanks to the use of a variational multiscale stabilized formulation), and that imposition of the normal velocity boundary condition is incorporated in the variational form. Equal-order interpolations can be quite useful, since Raviart–Thomas finite elements generate non-diagonal mass matrices which involve linear matrix solves in the time-integration procedure. Recently, mass lumping techniques have been proposed to improve efficiency in computations with Raviart–Thomas approximation spaces [4,11]. Needless to say that equal-order continuous finite elements allow for very standard mass lumping strategies when required for efficiency. Weak boundary conditions are particularly useful when considering complex geometry domains, in which the normal velocity component has to be enforced on curved surfaces. In this case it can be quite tedious to strongly constrain the components of the velocity at the nodes so that the normal velocity boundary condition is satisfied strongly.

In the case of nonlinear systems, the situation can become much more complicated when trying to strongly enforce the normal components of the velocity at the mesh nodes, due to additional algorithmic constraints. For example, while in linear problems discrete conservation is not always considered essential, in nonlinear equations it often gains a fundamental role. Considering Lagrangian shock flow problems (which require the same pressure and normal velocity boundary conditions as the wave propagation problems discussed here), the specific definition of normals at the nodes of the mesh has a direct consequence on whether or not the resulting numerical algorithm will be conservative. If conservation is the goal when using strong normal velocity boundary conditions, then one (if not the only) option possible is to evaluate the values of the surface normals at the mesh nodes by means of a global optimization problem with the conservation statement as a constraint. On the other hand, weak normal velocity boundary conditions guarantee conservation for these more complicated nonlinear systems with minimal implementation effort.

We also note that due to the particular hyperbolic structure of the problem under consideration, our formulation of weak boundary conditions *does not rely on penalty parameters*, in contrast to previous approaches targeting elliptic and parabolic equations [2,3]. In more complex nonlinear systems, this aspect is also of importance, since often times the appropriate choice of the penalty parameter may result in a tedious exercise, or even lead to somewhat unsatisfactory results.

The rest of the exposition is organized as follows: Section 2 is devoted to presenting the equations of propagation for longitudinal waves. The general variational setting is discussed in Section 3. In Section 4, we present the framework of weak boundary conditions and we also propose a specific numerical implementation using a variational multiscale method derived from the work in [61]. The results of numerical experiments to test and demonstrate the proposed boundary condition framework are analyzed in Section 5. Conclusions are summarized in Section 6.

2. General equations

The present work is focused on the propagation of waves in confined (bounded) domains. It is not our purpose to investigate issues related to unbounded domains or open, nonreflective boundary conditions [15–28,31,47,53]. We are considering the classical equations of wave propagation in mixed form, specific to materials for which the stress tensor is given by a pressure term. Let Ω be an open and bounded set in $\mathbb{R}^{n_d}$ (where n_d is the number of spatial dimensions) and consider the system of equations given by:

$$\rho\partial_t \boldsymbol{v} + \nabla_{\boldsymbol{x}} p = \rho \boldsymbol{b}, \tag{1}$$

$$\partial_t p + \rho c_s^2 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0. \tag{2}$$

Here, $\nabla_{\boldsymbol{x}}$ and $\nabla_{\boldsymbol{x}}\cdot$ are the gradient and divergence operators, ∂_t indicates derivation with respect to time, and $\boldsymbol{b}$ is a body force. These equations describe the propagation of disturbances in a medium of density and wave speed given by ρ and c_s, respectively. Complete specification of the problem requires initial conditions $p = p_0(x)$ and $\boldsymbol{v} = \boldsymbol{v}_0(x)$ at $t = 0$, and appropriate boundary conditions.

Assuming that the boundary $\Gamma = \partial\Omega$ is partitioned as $\overline{\Gamma_g \cup \Gamma_h}$, $\Gamma_g \cap \Gamma_h = \emptyset$, *pressure boundary conditions* are enforced on the *Dirichlet* boundary Γ_g, that is,

$$p|_{\Gamma_g} = g(\boldsymbol{x}, t), \tag{3}$$

and *normal velocity boundary conditions* are enforced on the *Neumann* boundary Γ_h,

$$\boldsymbol{v} \cdot \boldsymbol{n}|_{\Gamma_h} = h(\boldsymbol{x}, t), \tag{4}$$

where $\boldsymbol{n}$ is the outward-pointing normal.

Remark 1. In the context of longitudinal wave propagation in fluids the boundary condition setting is somewhat *reversed* with respect to the case of wave propagation in solids. In the latter case, velocity boundary conditions are of Dirichlet-type and stress boundary conditions are of Neumann type, since tangential waves are present in addition to longitudinal waves. This leads to the connection between waves in solids and the *Stokes flow problem*, and between longitudinal waves in fluids and the *Darcy flow problem*.

The structure of Eqs. (1) and (2) is common to many other mechanical systems, such as non-dissipative acoustics, gravity wave propagation in shallow water flows, simplified models for global circulation of meteorological flows, and more generally, the propagation of longitudinal waves in non-dissipative media. In the case of acoustics, these equations can be derived by linearization of the compressible Euler equations characterizing inviscid fluids, as shown in [64, p. 158] or also [61]. We will always consider the case when ρ and c_s are finite and positive. Denoting by $\zeta = \rho c_s$ the *characteristic impedance* of the medium, and introducing the parameter $\chi = \rho\,\zeta^{-2}$, we can rearrange the previous equations as follows:

$$\rho\partial_t \boldsymbol{v} + \nabla_{\boldsymbol{x}} p = \rho \boldsymbol{b}, \tag{5}$$

$$\chi\partial_t p + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0, \tag{6}$$

or, in vector operator form,

$$\left(\begin{bmatrix} \rho \boldsymbol{I}_{n_d\times n_d} & \boldsymbol{0}_{n_d\times 1} \\ \boldsymbol{0}_{1\times n_d} & \chi \end{bmatrix} \frac{\partial}{\partial t} + \begin{bmatrix} \boldsymbol{0}_{n_d\times n_d} & \nabla_{\boldsymbol{x}} \\ \nabla_{\boldsymbol{x}}\cdot & 0 \end{bmatrix} \right) \begin{Bmatrix} \boldsymbol{v} \\ p \end{Bmatrix} = \begin{Bmatrix} \rho \boldsymbol{b} \\ 0 \end{Bmatrix}. \tag{7}$$

In the case of constant material properties ρ and c_s and zero forcing term $\boldsymbol{b}$, it is not difficult to see that, by combining appropriate