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Algebraic techniques for eigenvalues and eigenvectors of a split quaternion matrix in split quaternionic mechanics[☆]

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ABSTRACT

This paper, by means of complex representation of a split quaternion matrix, studies the problems of right split quaternion eigenvalues and eigenvectors of a split quaternion matrix, and derives algebraic techniques for the right split quaternion eigenvalues and eigenvectors of the split quaternion matrix in split quaternionic mechanics.

1. Introduction

In the study of the relation between complexified classical and non-Hermitian quantum mechanics, physicians found that there are links to quaternionic and coquaternionic mechanics. The main finding is that complexified mechanical systems with real energies studied extensively in the literature over the past decade can alternatively be thought of as certain split quaternionic extensions of the underlying real mechanical systems. This identification leads to the possibility of employing algebraic techniques of quaternions and split quaternions to tackle some of the challenging open issues in complexified classical and quantum mechanics.

Complex (i.e. non-Hermitian) Hamiltonians have long been employed to describe open quantum systems, decay and scattering phenomena [1]. Further, since the realisation that complex operators respecting space–time reflection (PT) symmetry may possess entirely real spectra [2], there have been considerable research interests in examining both physical and mathematical properties of quantum systems described by non-Hermitian Hamiltonians with real spectra. More recently, the interest in these systems has increased notably, in part owing to experimental realisations of the phenomenon of the PT phase transition and other theoretically predicted effects [3,4].

Complexified classical mechanics has also been studied intensely both in the context of semiclassical calculations and as a classical analogue of non-Hermitian quantum mechanics [5–9]. For a classical system, its complex extension typically involves the use of complex phase-space variables, and the Hamiltonian in general also becomes complex. For a quantum system, on the other hand, its complex extension typically involves the use of a Hamiltonian that is not Hermitian, a fully complexied quantum system analogous to its classical counterpart, can be formulated, where state space variables are also complexied [10,11]. There are two natural ways in which quantum system can be extended into a fully complex domain [11], where both the Hamiltonian and the state space are complexied. In short, one is to let state space variables and Hamiltonian be quaternion valued; the other is to let them split quaternion valued. The former is related to those of symmetric quantum system of Finkelstein and others [12,13] which can be viewed as representing complex extensions of the underlying real mechanics for real Hermitian quantum systems, whereas the latter possesses spectral structures similar to those of PT-symmetric quantum system of Bender and others [7–10] which can be viewed as representing complex extensions of the underlying real mechanics for real non-Hermitian quantum systems.

The statements above suggest that complexified quantum mechanics can alternatively be viewed as a version of quaternionic or split quaternionic quantum mechanics, and PT-symmetric systems are in fact related to split quaternions, rather than quaternions. In this connection it is worth remarking that symmetries of split quaternion are related to the Lorentz group, rather than Euclidean

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group, in the sense that every rotation in the Minkowski three-space can be expressed in terms of split quaternions. Hence PT symmetry a priori has more in common with Lorentzian space–time than Euclidean space–time. It follows that PT symmetric quantum mechanics is not equivalent to the traditional quaternionic quantum theories, instead it is equivalent to a split quaternionic quantum mechanics.

A split quaternion (or coquaternion), which was found in 1849 by James Cockle, is in the form of

$$q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k},$$
$$\mathrm{i}^2 = -1, \mathrm{j}^2 = \mathrm{k}^2 = 1, \mathrm{ijk} = 1,$$

where q_0, q_1, q_2, q_3 are real numbers. One can see easily that $\mathrm{ij} = -\mathrm{ji} = \mathrm{k}, \mathrm{jk} = -\mathrm{kj} = -\mathrm{i}, \mathrm{ki} = -\mathrm{ik} = \mathrm{j}$. Denote the sets of split quaternions and quaternions respectively by $\mathbf{H}_s$ and $\mathbf{H}$. The set $\mathbf{H}_s$ of split quaternions is an associative and non-commutative 4-dimensional Clifford algebra, and it contains zero divisors, nilpotent elements and nontrivial idempotents [14–17]. The ring $\mathbf{H}_s$ and the quaternion ring $\mathbf{H}$ are two different non-commutative 4-dimensional Clifford algebra, the ring $\mathbf{H}$ is a skew-field, and the ring $\mathbf{H}_s$ is not. The structure of ring $\mathbf{H}_s$ is more complicated than that of the ring $\mathbf{H}$.

A split quaternion λ is said to be a right (left) eigenvalue provided that $A\alpha = \alpha\lambda (A\alpha = \lambda\alpha)$ for nonzero vector α, and α is said to be an eigenvector to the right (left) eigenvalue λ. The eigen-problem of quaternion matrices and split quaternion matrices play important roles in the study of theories and numerical computations of quaternionic and split quaternionic mechanics. There are a lot of works associated with quaternionic eigenvalue problem [18–25]. In papers [18,19], we studied the problems of right eigenvalues and eigenvectors of quaternion matrices by means of complex representation of a quaternion matrix, derived algebraic techniques for the right eigenvalues and eigenvectors of the quaternion matrices in quaternionic quantum mechanics. In paper [26], the authors discussed the properties of complex eigenvalues of a split quaternion matrix, and gave an extension of Gershgorin theorem. In paper [11], the authors studied the properties and applications of 2×2 split quaternionic Hermitian matrices, and obtained that the eigenvalues were either real or appeared as complex conjugate pairs. In general the following problems have hitherto remained tangential for a split quaternion matrix A in split quaternionic mechanics.

Problem 1. Does it exists right split quaternion eigenvalues for A? What is a necessary and sufficient conditions for A to have a right split quaternion eigenvalue?

Problem 2. How to find all possible right split quaternion eigenvalue and corresponding split quaternion eigenvectors of A?

This paper, by means of complex representation of a split quaternion matrix, studies the eigen-problem of right split quaternion eigenvalues and corresponding split quaternion eigenvectors of a split quaternion matrix, and settles down the two problems above. It not only gives a necessary and sufficient conditions for A to have a right split quaternion eigenvalue, but also derives algebraic techniques for the right split quaternion eigenvalues and corresponding split quaternion eigenvectors of the split quaternion matrix in split quaternionic mechanics.

Let $\mathbf{R}$ be the real number field, $\mathbf{C} = \mathbf{R} \oplus \mathbf{R}\mathrm{i}$ the complex number field, and $\mathbf{H}_s = \mathbf{R} \oplus \mathbf{R}\mathrm{i} \oplus \mathbf{R}\mathrm{j} \oplus \mathbf{R}\mathrm{k}$ the split quaternion ring, where $\mathrm{i}^2 = -1, \mathrm{j}^2 = \mathrm{k}^2 = 1, \mathrm{ijk} = 1$. $\mathrm{F}^{m\times n}$ denotes the set of $m \times n$ matrices on a ring F. If $q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k} \in \mathbf{H_s}$, where $q_0, q_1, q_2, q_3 \in \mathbf{R}$, then define $\overline{q} = q_0 - q_1\mathrm{i} - q_2\mathrm{j} - q_3\mathrm{k}$ to be conjugate of q. Define $\mathrm{Re}(q) = q_0$, the real part of $\mathrm{Im}(q) = q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}$, the imaginary part of q. The norm $|q|$ of a split quaternion q is defined as $|q| = \sqrt{|q\overline{q}|} = \sqrt{|q_0^2 + q_1^2 - q_3^2 - q_4^2|}$. q is said to be a unit split quaternion if its norm is 1. If $A \in \mathbf{H}_s^{m\times n}$, let $A = A_0 + A_1\mathrm{i} + A_2\mathrm{j} + A_3\mathrm{k}, A_t \in \mathbf{R}^{m\times n}$ define $\overline{A} = A_0 - A_1\mathrm{i} - A_2\mathrm{j} - A_3\mathrm{k}$ to be conjugate of A. In addition, define other conjugates for a split quaternion matrix $A = A_0 + A_1\mathrm{i} + A_2\mathrm{j} + A_3\mathrm{k} \in \mathbf{H}_s^{m\times n}$ respectively to be as follows.

$$\begin{aligned} A^{(12)} &= A^{(21)} = A_0 - A_1\mathrm{i} - A_2\mathrm{j} + A_3\mathrm{k}, \\ A^{(13)} &= A^{(31)} = A_0 - A_1\mathrm{i} + A_2\mathrm{j} - A_3\mathrm{k}, \end{aligned} \tag{1.1}$$

$$\begin{aligned} A^{(23)} &= A^{(32)} = A_0 + A_1\mathrm{i} - A_2\mathrm{j} - A_3\mathrm{k}, \\ A^{(123)} &= A^{(321)} = A_0 - A_1\mathrm{i} - A_2\mathrm{j} - A_3\mathrm{k}, \end{aligned} \tag{1.2}$$

the matrix $A^{(st)}$ is called to be (st)-conjugate of the matrix A, $s \neq t, s, t = 1, 2, 3$. Clearly, $A^{(12)} = A^{(21)} = \mathrm{k}A\mathrm{k}, A^{(13)} = A^{(31)} = \mathrm{j}A\mathrm{j}$, $A^{(23)} = A^{(32)} = -\mathrm{i}A\mathrm{i}$, and $A^{(123)} = A^{(321)} = \overline{A}$. Clearly $(A + B)^{(123)} = A^{(123)} + B^{(123)}, (AB)^{(123)} \neq A^{(123)}B^{(123)}$ in general. The (12)-conjugate, (13)-conjugate and (123)-conjugate are three different generations of ordinary conjugate of a complex matrix. It is easy to get following equations by the definitions above for any $A, B \in \mathbf{H}_s^{m\times n}, C \in \mathbf{H}_s^{n\times p}$,

$$(A + B)^{(st)} = A^{(st)} + B^{(st)}, \quad (AC)^{(st)} = A^{(st)}C^{(st)}, \tag{1.3}$$

in which $s \neq t, s, t = 1, 2, 3$.

For $A \in \mathbf{C}^{m\times m}$, $f_A(x)$ denotes the characteristic polynomial of A.

Lemma 1.1 ([27]). *Let $A \in \mathbf{C}^{m\times m}$, $B \in \mathbf{C}^{n\times n}$, $C \in \mathbf{C}^{m\times n}$. Then the matrix equation $AX - XB = C$ has a unique solution $X \in \mathbf{C}^{m\times n}$ if and only if $f_A(x)$ and $f_B(x)$ are relatively prime, i.e. $(f_A(x), f_B(x)) = 1$, in other words, $f_A(B)$ is a nonsingular matrix.*

2. Split quaternions and equivalence classes

Two split quaternions p and q are said to be similar if there exists a nonsingular split quaternion x such that $x^{-1}px = q$; this is written as $p \sim q$. Obviously, p and q are similar if and only if there is a unit split quaternion u such that $u^{-1}pu = q$, and two similar split quaternions have the same norm. Clearly, $\sim$ is an equivalence relation on the split quaternions. We denote by $[q]$ the equivalence class containing the split quaternion q. The split quaternion q is called to be a principal element of the equivalence class $[q]$.

Proposition 2.1 ([22]). *Let $p = p_0 + p_1\mathrm{i} + p_2\mathrm{j} + p_3\mathrm{k}$ and $q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}$ be two split quaternions. Then p and q are similar if and only if $p_0 = q_0$ and $p_1^2 - p_2^2 - p_3^2 = q_1^2 - q_2^2 - q_3^2$, i.e. $\mathrm{Re}(q) = \mathrm{Re}(q)$, $|\mathrm{Im}(p)| = |\mathrm{Im}(q)|$.*

For any split quaternion $q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k} = a + b\mathrm{j}$, $a = q_0 + q_1\mathrm{i}, b = q_2 + q_3\mathrm{i}, q_0, q_1, q_2, q_3 \in \mathbf{R}$, the complex representation of q is defined [14] to be

$$q^C = \begin{bmatrix} a & b \\ \overline{b} & \overline{a} \end{bmatrix} \in \mathbf{C}^{2\times 2} \tag{2.1}$$

and by [14], for any two split quaternions p and q, we have that $(p + q)^C = p^C + q^C, (pq)^C = p^C q^C$. The characteristic polynomial of matrix q^C is $f_{q^C}(x) = x^2 - (a + \overline{a})x + a\overline{a} - b\overline{b}$. It is easy to know that the two complex eigenvalues of the complex representation q^C are $\lambda_1 = q_0 + \sqrt{q_2^2 + q_3^2 - q_1^2}, \lambda_2 = q_0 - \sqrt{q_2^2 + q_3^2 - q_1^2}$.

Case 1: If $q_1^2 > q_2^2 + q_3^2$, the two different imaginary eigenvalues are $\lambda = q_0 + \sqrt{q_1^2 - q_2^2 - q_3^2}\mathrm{i}, \overline{\lambda} = q_0 - \sqrt{q_1^2 - q_2^2 - q_3^2}\mathrm{i}$, and clearly there exists a nonsingular complex matrix T_1 such that $T_1 = x_1^C$, and

$$q^C T_1 = T_1 \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix} \Leftrightarrow qx_1 = x_1\lambda, \tag{2.2}$$