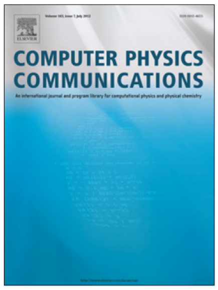

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Positivity-preserving scheme for two-dimensional advection-diffusion equations including mixed derivatives

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Abstract

In this work, we propose a positivity-preserving scheme for solving two-dimensional advection-diffusion equations including mixed derivative terms, in order to improve the accuracy of lower-order methods. The solution of these equations, in the absence of mixed derivatives, has been studied in detail, while positivity-preserving schemes for mixed derivative terms have received much less attention. A two-dimensional diffusion equation, for which the analytical solution is known, is solved numerically to show the applicability of the scheme. It is further applied to the Fokker-Planck collision operator in two-dimensional cylindrical coordinates under the assumption of local thermal equilibrium. For a thermal equilibration problem, it is shown that the scheme conserves particle number and energy, while the preservation of positivity is ensured and the steady-state solution is the Maxwellian distribution.
Keywords: Advection-diffusion, Fokker-Planck equation, low-order positivity-preserving scheme

1. Introduction

Two dimensional advection-diffusion equations have widespread applications in physics, engineering and finance, and can generally be written as

$$u_t = Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu \tag{1}$$

where $u = u(x, y, t)$. As these equations are often too difficult to solve analytically, numerical solutions are required. For $F = 0$, these equations can be written in a two-dimensional advection-diffusion form,

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(-\vec{a}u + \hat{k} \cdot \nabla u \right) \tag{2}$$

where $u = u(x, y, t)$ is advected by the 2D vector $\vec{a}(x, y, t)$ and diffused by the tensor $\hat{k}(x, y, t)$. A particular application of two-dimensional advection-diffusion equations is the Fokker-Planck collision operator, which can be written in the form (2) and has a wide range of applications in plasmas in the laboratory (e.g. magnetic and inertial thermonuclear fusion), space (e.g. Earth's magnetosphere), and astrophysics (e.g. solar coronal mass ejections) [1].

If the initial condition $u(x, y, 0) \geq 0$ holds for all (x, y), then the solution must always be positive, i.e. $u(x, y, t) \geq 0$ for all (x, y, t) when $F = 0$. Moreover, a good numerical method has to preserve the monotonicity of the initial condition. The conservation of such properties poses a particular challenge if a change of coordinates, adopted to eliminate the mixed derivative terms u_{xy} throughout (x, y) space, is not possible.

Positivity-preserving schemes for two-dimensional advection-diffusion equations, in the absence of mixed derivative terms, have been studied in detail [2, 3], but schemes for problems where mixed derivative terms are present have received much less attention. Recent research have focused on developing improved and refined higher-order methods for solving advection-diffusion equations, as these methods have the potential of providing accurate solutions at reasonable cost. Implementing these methods, however, can be complicated. In contrast, lower-order methods, while being less accurate, are easier to implement and generally more robust and reliable, and are therefore routinely employed in practical calculations [4].

In this paper, we propose a scheme for improving the accuracy of lower-order methods, in particular with respect to the preservation of positivity, when solving two-dimensional advection-diffusion equations in the presence of mixed derivatives, as lower-order methods are routinely used for solving these equations, yet fail to preserve

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