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A time-spectral approach to numerical weather prediction ☆

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ABSTRACT

Finite difference methods are traditionally used for modelling the time domain in numerical weather prediction (NWP). Time-spectral solution is an attractive alternative for reasons of accuracy and efficiency and because time step limitations associated with causal CFL-like criteria, typical for explicit finite difference methods, are avoided. In this work, the Lorenz 1984 chaotic equations are solved using the time-spectral algorithm GWRM (Generalized Weighted Residual Method). Comparisons of accuracy and efficiency are carried out for both explicit and implicit time-stepping algorithms. It is found that the efficiency of the GWRM compares well with these methods, in particular at high accuracy. For perturbative scenarios, the GWRM was found to be as much as four times faster than the finite difference methods. A primary reason is that the GWRM time intervals typically are two orders of magnitude larger than those of the finite difference methods. The GWRM has the additional advantage to produce analytical solutions in the form of Chebyshev series expansions. The results are encouraging for pursuing further studies, including spatial dependence, of the relevance of time-spectral methods for NWP modelling.

Program summary
Program Title: Time-adaptive GWRM Lorenz 1984
Program Files doi: http://dx.doi.org/10.17632/4nxfyjj7nv.1
Licensing provisions: MIT
Programming language: Maple
Nature of problem: Ordinary differential equations with varying degrees of complexity are routinely solved with numerical methods. The set of ODEs pertaining to chaotic systems, for instance those related to numerical weather prediction (NWP) models, are highly sensitive to initial conditions and unwanted errors. To accurately solve ODEs such as the Lorenz equations (E. N. Lorenz, Tellus A 36 (1984) 98–110), small time steps are required by traditional time-stepping methods, which can be a limiting factor regarding the efficiency, accuracy, and stability of the computations.
Solution method: The Generalized Weighted Residual Method, being a time-spectral algorithm, seeks to increase the time intervals in the computation without degrading the efficiency, accuracy, and stability. It does this by postulating a solution ansatz as a sum of weighted Chebyshev polynomials, in combination with the Galerkin method, to create a set of linear/non-linear algebraic equations. These algebraic equations are then solved iteratively using a Semi Implicit Root solver (SIR), which has been chosen due to its enhanced global convergence properties. Furthermore, to achieve a desired accuracy across the entire domain, a time-adaptive algorithm has been developed. By evaluating the magnitudes of the Chebyshev coefficients in the time dimension of the solution ansatz, the time interval can either be decreased or increased.

1. Introduction

1.1. Lorenz 1984 model

The study of atmospheric dynamics is challenging due to its complex and chaotic nature. Lorenz [1] proposed in 1984 a simplified model for representing Hadley circulation of air in the

☆ This paper and its associated computer program are available via the Computer Physics Communication homepage on ScienceDirect (http://www.sciencedirect.com/science/journal/00104655).

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atmosphere:

$$\frac{dX}{dt} = -Y^2 - Z^2 - aX + aF \quad (1)$$

$$\frac{dY}{dt} = XY - bXZ - Y + G \quad (2)$$

$$\frac{dZ}{dt} = bXY + XZ - Z \quad (3)$$

The equations are, in his words, "the simplest model capable of representing an unmodified or modified Hadley circulation, determining its stability, and, if it is unstable, representing a stationary or migratory disturbance" [1]. Lorenz uses the variable X to represent the intensity of the symmetric globe-encircling westerly wind current, and also the poleward temperature gradient. The variables Y and Z represent the cosine and sine phase of a chain of superposed large-scale eddies. The parameters a, b, F and G may be chosen within certain bounds.

The model demonstrates chaotic behaviour for certain sets of parameters. In want of analytical solutions, initial-value problems are traditionally solved purely numerically by the use of finite steps in the temporal domain. The time steps of explicit time advance methods for general, space-dependent problems are restricted to small values through constraints such as the Courant–Friedrichs–Lewy (CFL) condition, and implicit schemes require time-consuming matrix operations at each time step. Semi-implicit methods allow large time steps and more efficient matrix inversions than those of implicit methods, but may feature limited accuracy. In this work we suggest an alternative, time-spectral approach for solution of Eqs. (1)–(3).

1.2. Generalized Weighted Residual Method

The Generalized Weighted Residual Method (GWRM) differs from traditional spectral methods for initial-value problems [2] in that also the time domain is treated spectrally [3,4]. As a result the GWRM eliminates grid causality conditions such as the CFL condition, being associated with time stepping algorithms. Although the problems to be solved are assumed causal, the method is acausal in the sense that the time dependence is calculated by a global minimization procedure (the weighted residual formalism) acting on the time integrated problem. Recall that in standard WRM [5], initial value problems are transformed into a set of coupled ordinary, linear or non-linear, differential equations for the time-dependent expansion coefficients. These are solved using finite differencing techniques.

The GWRM enables semi-analytical solutions; finite approximate solutions are obtained as analytical Chebyshev expansions. Not only temporal and spatial, but also physical parameter domains may be treated spectrally, being of interest for carrying out scaling dependence in a single computation. Chebyshev polynomials are used for the spectral representation. These have several desirable qualities. They converge rapidly to the approximated function, they are real and can straightforwardly be converted to ordinary polynomials and vice versa, their minimax property guarantees that they are the most economical polynomial representation, they can be used for non-periodic boundary conditions (being problematic for Fourier representations) and they are particularly apt for representing boundary layers where their extrema are locally dense [3].

In standard text books on spectral methods for differential equations, time-spectral methods are usually touched upon only briefly and dismissed on the grounds that they are expensive [6,7]. Historically, a number of authors have investigated various suggestions for and aspects of spectral methods in time. In 1979 a pseudo-spectral method, based on iterative calculation and an approximate factorization of the given equations, was suggested in [8]. Also, some early ideas were not developed further by Peyret and Taylor in [9].

In 1986 and 1989, Tal-Ezer [10,11] proposed time-spectral methods for linear, periodic hyperbolic and parabolic equations, respectively, using a polynomial approximation of the evolution operator in a Chebyshev least square sense. Periodicity was assumed for the spatial domain through use of the Fourier spectral approximation. The method extends the traditional $\Delta t = O(1/N^2)$ stability criterion for explicit algorithms, where the space resolution parameter $N = O(1/\Delta x)$, to higher efficiency resulting in the stability condition $\Delta t = O(1/N)$). This approach to extend the time step in explicit methods was further studied in [12]. The method is not widely used; a reason for this may be its complexity and its restriction to certain classes of problems. Later, Luo extended the method to more general boundary conditions and multiple spatial dimensions [13].

Ierley et al. [14] solved a class of nonlinear parabolic partial differential equations with periodic boundary conditions using a Fourier representation in space and a Chebyshev representation in time. Similarly as for the GWRM, the Burger equation and other problems were solved with high resolution. Tang and Ma [15] also used a spatial Fourier representation for solution of parabolic equations, but introduced Legendre Petrov–Galerkin methods for the temporal domain.

In 1994, Bar-Yoseph et al. [16,17] used space–time spectral element methods for solving one-dimensional nonlinear advection–diffusion problems and second order hyperbolic equations. Chebyshev polynomials were later employed in space–time least-squares spectral element methods [18].

A theoretical analysis of Chebyshev solution expansion in time and one-dimensional space, for equal spectral orders, was given in [19]. The minimized residuals employed were however different from those of the GWRM.

More recently Dehghan and Taleei [20] found solutions to the non-linear Schrödinger equation, using a time–space pseudo-spectral method where the basis functions in time and space were constructed as a set of Lagrange interpolants.

Time-spectral methods feature high order accuracy in time. For implicit finite difference methods, deferred correction may provide high order temporal accuracy [21,22]. A relatively recent approach to increase the temporal efficiency of finite difference methods is time-parallelization via the parareal algorithm [23]. This method, however, features rather low parallel efficiency and improvements have been suggested, for example the use of spectral deferred corrections [24].

An interesting Jacobian-free Newton–Krylov method for implicit time-spectral solution of the compressible Navier–Stokes equations has recently been put forth by Attar [25].

A time-spectral method for periodic unsteady computations, using a Fourier representation in time, was suggested in [26] and further developed in [27] and [28]. A generalization to quasi-periodic problems was developed in [29].

In summary, although time-spectral methods have been explored in various forms by several authors during the last few decades, and were found to be highly accurate, the GWRM as described in [3] has not been pursued. The present work contributes to the evaluation of this method.

The structure of the paper is as follows. In Section 2 we briefly review the general GWRM formalism for solving a set of PDEs but subsequently restrict us to a discussion of an optimized solution of the ODEs (1)–(3). A major goal of this study is to evaluate possible advantages of the GWRM in relation to finite difference methods (FDM). Thus a comparative numerical study of convergence, accuracy, and efficiency for short time interval solutions are presented in Section 3. We have selected the explicit fourth order Runge–Kutta method (RK4) to represent efficient FDM. Comparisons have also be made with well-known implicit methods;