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Numerical computation of electromagnetic field for general static and axisymmetric current distribution

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ABSTRACT

We developed a numerical method to compute the electromagnetic field of arbitrary static and axisymmetric current distribution. The method (i) numerically evaluates a double integral of the electrostatic and magnetostatic potentials of an infinitely thin ring current by the split quadrature method using the double exponential rules, and (ii) derives the electrostatic field and the magnetostatic induction by numerically differentiating the numerically integrated potentials by the central difference formula. A comparison with the exact solution for a poloidal current distribution with an anisotropic Gaussian damping confirmed the 14- and 9-digit accuracy of the potential and the field/induction computed by the new method.

1. Introduction

The computation of the electromagnetic field for a general axisymmetric three-dimensional charge/current distribution is a classic problem in physics and engineering [1,2]. Indeed, its applications are as wide as (i) the electron and ion optics [3], (ii) the charged particle acceleration [4], (iii) the electron microscopy and spectroscopy [5], and (iv) the magnetic coil design [6]. Especially, it is one of the building blocks of the plasma physics and controlled nuclear fusion [7,8].

If the spatial distribution of the static electric charge, $\rho(\mathbf{x})$, and of the static current vector, $\mathbf{J}(\mathbf{x})$, are explicitly known, then the electrostatic scalar potential, $\Phi(\mathbf{x})$, and the magnetostatic vector potential, $\mathbf{A}(\mathbf{x})$, are written as convolutions of these distributions with the Newton kernel, $1/|\mathbf{x}-\mathbf{x}'|$, [9, equations (1.17) and (5.32)] as

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho\left(\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3\mathbf{x}', \tag{1}$$

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}\left(\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3\mathbf{x}', \tag{2}$$

where the integration is conducted over all the volume occupied by the charge and/or the current vector, and ε_0 and μ_0 are the vacuum permittivity and permeability, respectively. The associated electrostatic field and the resulting magnetostatic induction are expressed as

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho\left(\mathbf{x}'\right)\frac{\left(\mathbf{x}-\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|^3}\mathrm{d}^3\mathbf{x}', \tag{3}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \mathbf{J}\left(\mathbf{x}'\right)\times\frac{\left(\mathbf{x}-\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|^3}\mathrm{d}^3\mathbf{x}'. \tag{4}$$

These are nothing but Coulomb's law and the Biot–Savart law [9, equations (1.5) and (5.14)].

When the charge/current distribution is finitely bounded, the external electromagnetic field can be expanded in harmonics [10]. However, if the evaluation point $\mathbf{x}$ is inside the distributions of the charge or current, on the other hand, the integral expressions, Eqs. (1)–(4), suffer from the algebraic singularities. This becomes a serious issue for extended distributions such as encountered in the plasma physics.

Before going further, let us show a practical example. Fig. 1 shows the contour map on a meridional cross section of a hypothetical charge/current distribution. It was designed to resemble the poloidal mode equilibrium solution of the plasma current distribution circulating in an ITER-like tokamak [11, Fig. 4]. Refer to Section 4 later for the detailed model description.

Although the adopted model distribution is infinitely extended in principle, it can be practically regarded to be finitely bounded thanks to the Gaussian damping around the central ring. Inside this practical boundary, the algebraic singularities appear everywhere. Thus, $\mathbf{E}(\mathbf{x})$ and/or $\mathbf{B}(\mathbf{x})$ are hardly computed by evaluating the integral forms by the existing quadrature techniques [12].

Therefore, a common practice has been solving Poisson's equation for $\Phi(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ [9, equations (1.28) and (5.28)], which are

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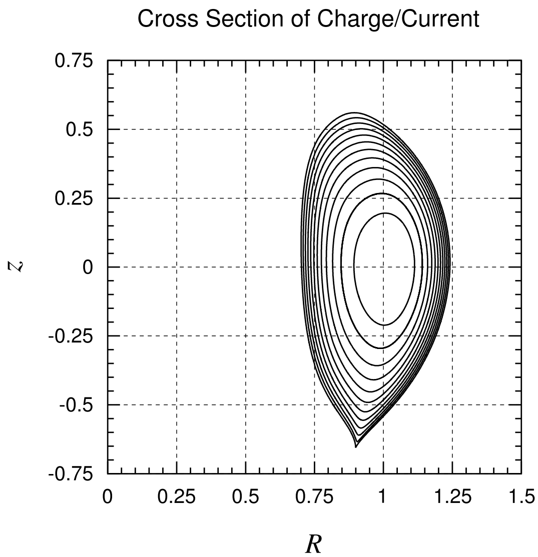

Fig. 1. Cross section of model electric charge/current distribution. Shown is the contour map on the meridional cross section of a hypothetical electric charge/current distribution. The contours are drawn for the levels of the relative magnitude being inverse powers of 2 as $\rho/\rho_0 = J/J_0 = 2^{-n}$ for $n = 1, 2, \ldots, 12$. Although the distribution is infinitely extended, it is practically bounded in a finite region thanks to the Gaussian damping feature adopted in the model distribution.

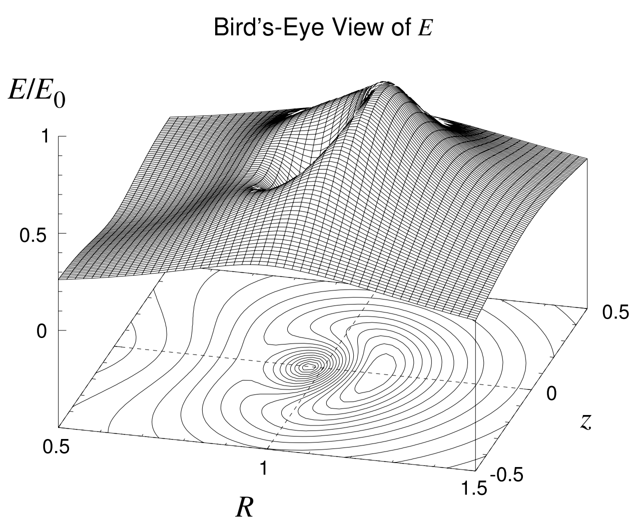

Fig. 2. Bird's-eye view of electrostatic field strength. Displayed is a bird's-eye view of $E \equiv |\mathbf{E}(\mathbf{x})|$, the magnitude of the electrostatic field caused by the current distribution depicted in Fig. 1.

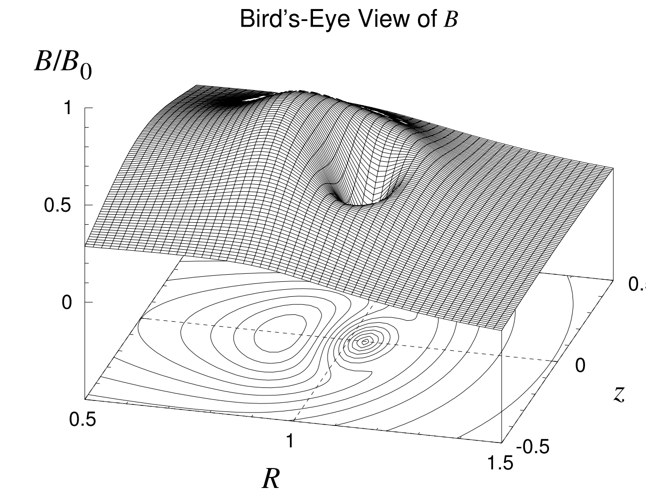

Fig. 3. Bird's-eye view of magnetostatic field strength. Displayed is a bird's-eye view of $B \equiv |\mathbf{B}(\mathbf{x})|$, the magnitude of the magnetostatic induction caused by the current distribution depicted in Fig. 1.

nothing but the differential form of Gauss' and Ampere's laws, respectively:

$$\nabla^2 \Phi = -\rho/\varepsilon_0, \tag{5}$$

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}. \tag{6}$$

These equations are elliptic type partial differential equations. They are numerically solved by the finite or boundary element methods of various kinds [13–17]. Refer to Bellina and Serra [18] for a concise summary of the numerical approaches. Nonetheless, the resulting formulation becomes cumbersome in general [19] and suffers from the accuracy degrade [20]. This is especially true if the boundary conditions are complicated [21–23].

Recently, we developed a numerical method to circumvent the difficulties for the gravitational field of an axisymmetric mass density distribution [24]. It can be directly applicable to the computation of $\Phi(\mathbf{x})$ and $\mathbf{E}(\mathbf{x})$ as summarized in Appendix A. Therefore, in this article, we adapt the method to the computation of $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ for arbitrary axisymmetric distribution of electric current. By using the original and adapted methods, we prepared Figs. 2 and 3 showing the bird's-eye views of $\mathbf{E}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ of the hypothetical charge/current distribution specified in Fig. 1. As will be shown later, these results are of the 9-digit accuracy, which is far more than necessary.

Below, we (i) describe the adapted method in Section 2, (ii) examine its computational accuracy and speed in Section 3, and (iii) present its example in Section 4.

2. Method

Consider a general static and axisymmetric current distribution. Adopt the cylindrical polar coordinate system, (R, z, ϕ). In this case, the only non-zero components of $\mathbf{J}(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ are their azimuthal components:

$$J(R, z) \equiv J_\phi(R, z), \quad A(R, z) \equiv A_\phi(R, z). \tag{7}$$

By symmetry, $\mathbf{A}(\mathbf{x})$ vanishes on the z-axis as

$$A(0, z) = 0. \tag{8}$$

Therefore, we scale it as

$$a(R, z) \equiv A(R, z)/R. \tag{9}$$

Denote the lower and upper end point of the radial distribution by $R_{\rm L}(\geq 0)$ and $R_{\rm U}(\leq +\infty)$, respectively. For simplicity, we assume that $J(R, z)$ vanishes when $z \leq z_{\rm L}(R)$ or $z \geq z_{\rm U}(R)$ where $z_{\rm L}(R)(\geq -\infty)$ and $z_{\rm U}(R)(\leq +\infty)$ are certain functions of R. Then, $a(R, z)$ is expressed as a double integral convolving $J(R, z)$ with Green's function as

$$a(R, z) = \int_{R_{\rm L}}^{R_{\rm U}} \left(\int_{z_{\rm L}(R')}^{z_{\rm U}(R')} F\left(R', z'; R, z\right) {\rm d}z' \right) {\rm d}R', \tag{10}$$

where we abbreviate the integrand as

$$F\left(R', z'; R, z\right) \equiv f\left(R', z'; R, z\right) J\left(R', z'\right), \tag{11}$$