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New energy-preserving schemes using Hamiltonian Boundary Value and Fourier pseudospectral methods for the numerical solution of the "good" Boussinesq equation

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ABSTRACT

Two energy-preserving schemes are proposed for the "good" Boussinesq (GBq) equation using the Hamiltonian Boundary Value and Fourier pseudospectral methods. The equation is discretized in space by Fourier pseudospectral method and in time by Hamiltonian Boundary Value methods (HBVMs). The outstanding advantages of the proposed schemes are that they can precisely conserve the global mass and energy, and provide highly accurate results. The single solitary wave, the interaction of two solitary waves and the birth of solitary waves are presented to validate the accuracy and conservation properties of the proposed schemes. In addition, we also compare our numerical results with other known studied methods in terms of numerical accuracy and conservation properties.

1. Introduction

Nonlinear partial differential equations have attracted much attention in studying evolution equations describing wave propagation. The Boussinesq (Bq) equation is one of the important models describing the nonlinear dispersive waves, which has many applications in many areas, e.g., ion-acoustic wave in plasma, magnetohydrodynamics wave in plasma, longitudinal dispersive wave in elastic rods and pressure wave in liquid–gas bubble mixtures, and so on. The Bq equation was first introduced by Boussinesq [1] in 1872. In this paper, we consider the following initial boundary value problem:

$$u_{tt} = u_{xx} - u_{xxxx} + (u^2)_{xx}, \quad (x, t) \in [a, b] \times [0, T], \tag{1.1}$$

with the initial condition

$$\begin{cases} u(x, 0) = \varphi_1(x), \\ \dfrac{\partial u}{\partial t}(x, 0) = \varphi_2(x), \quad x \in [a, b], \end{cases} \tag{1.2}$$

and the periodic boundary conditions

$$\begin{cases} u(a, t) = u(b, t), \\ \dfrac{\partial u}{\partial x}(a, t) = \dfrac{\partial u}{\partial x}(b, t), \quad t \in [0, T]. \end{cases} \tag{1.3}$$

In the literature, Eq. (1.1) is often known as GBq equation. In this paper, we consider the GBq equation (1.1), with a periodic boundary condition over a one-dimensional (1D) domain $\Omega = [a, b]$, and initial data $u(x, 0) = \varphi_1(x)$, $u_t(x, 0) = \varphi_2(x)$, both of which are periodic. It is assumed that unique, periodic, smooth enough solution exists for Eq. (1.1) over the time interval $[0, T]$. This periodicity assumption is reasonable if the solution to Eq. (1.1) decays exponentially outside $[a, b]$ [2]. Under the periodic boundary conditions, Eq. (1.1) has the following conserved invariants, i.e., mass and energy [3,4]

$$M = \int_a^b u dx, \tag{1.4}$$

$$E = \frac{1}{2} \int_a^b \left(v^2 + u^2 + \frac{2}{3} u^3 + u_x^2 \right) dx, \tag{1.5}$$

where $u_t = v_x$.

In recent years, along with the development of computer science, the study of the preservation of invariant tori for nearly integrable Hamiltonian systems has been a central theme in the research. With respect to this topic, researchers have proposed many methods, for example, symplectic (and/or symmetric) methods [5], however, these methods can only assure, at most, the conservation of quadratic Hamiltonian functions. On the other hand, it is possible to follow different approaches to derive geometric integrators which are energy-preserving. This

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has been done in the pioneering work [6], and later in [7], where discrete gradient methods are introduced and studied. In addition, another energy-preserving method, the Averaged Vector Field (AVF) method, which is considered in [8].

In this paper, the proposed schemes are constructed based on the HBVMs, which have been developed in a series of papers [9–23]. These methods, can be interpreted as an extension of the well-known Gauss–Legendre methods with the difference that HBVMs provide a precise energy conservation for polynomial Hamiltonian functions of any high degree. Moreover, Brugnano, Iavernaro and Trigiante [15] have proposed that these methods turn out to be symmetric, precisely A-stable, and can have arbitrarily high order. To our knowledge, the HBVMs is often used to solve ODEs, and there are few literatures about solving partial differential equations (PDEs). Nevertheless, Brugnano, Frasca Caccia and Iavernaro [17,18,24] have investigated the numerical solution of Hamiltonian PDEs using the energy-conserving methods in the HBVMs class, when a finite difference or spectral space discretization is considered. Frasca Caccia [23] has introduced the implementation and application of HBVMs.

Li and Vu-Quoc [25] once said that "in some areas, the ability to preserve some invariant properties of the original differential equation is a criterion to judge the success of a numerical simulation". Zhang, Víctor and Luis [26] pointed out that nonconservative schemes may easily show nonlinear blow-up. Thus, the main purpose of this paper is to study conservative schemes for the GBq equation. The GBq equation and its various extensions have been investigated by many authors. For instance, Manoranjan, Mitchell and Morris [3] presented a new solution for the two-soliton interaction of Eq. (1.1) and verified it using Galerkin methods. Ortega and Sanz-Serna [27] studied the nonlinear stability and convergence of some simple finite-difference schemes for Eq. (1.1). Frutos, Ortega and Sanz-Serna [28] proposed a pseudospectral scheme and studied its nonlinear stability and convergence for Eq. (1.1). Aydın and Karasözen [29] constructed second-order symplectic and multisymplectic integrators for Eq. (1.1) using the two-stage Lobatto IIIA–IIIB partitioned Runge–Kutta method. Mohebbi and Asgari [30] proposed three fast and high accuracy numerical methods for the problem (1.1). Cai and Wang [31] proposed a series of local structure-preserving algorithms for the GBq equation. Wang et al. [32] proposed an energy-preserving finite volume element method for the improved Boussinesq equation. More analytical and numerical works related to GBq equation can be found in the literature, for example, [2,33–41].

The outline of this paper is as follows. In Section 2, we present the framework of HBVMs. Then, in Section 3, we discretize the GBq equation with Fourier pseudospectral method in space, and derive the proposed energy-preserving schemes. All numerical experiments are presented in Section 4. The final section presents some simple conclusion.

2. Notation and preliminaries

In this section, we provide a novel framework for HBVMs. Here we briefly introduce the background information concerning the HBVMs. We follow the standard HBVMs formulation reported in [14].

Let

$$y' = J\nabla H(y), \qquad y(0) = y_0 \in \mathbb{R}^{2m}, \tag{2.1}$$

be a Hamiltonian problem in canonical form, where $J^T = -J = J^{-1}$ is a constant, orthogonal and skew-symmetric matrix, usually given by

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{2.2}$$

where I is the identity matrix of dimension m.

The scalar function $H(y)$ is the Hamiltonian of the problem and its value is constant during the motion, namely

$$H(y(t)) \equiv H(y_0), \quad \forall t \geq 0,$$

for the solution of (2.1). Indeed, one has:

$$\begin{aligned} \frac{d}{dt}H(y(t)) &= \nabla H(y(t))^T y'(t) \\ &= \nabla H(y(t))^T J \nabla H(y(t)) = 0, \quad \forall t \geq 0. \end{aligned} \tag{2.3}$$

Often, the Hamiltonian H is also called the energy, since for isolated mechanical systems it has the physical meaning of total energy. Consequently, energy conservation is an important feature in the simulation of such problems. On the other hand, assume that, in problem (2.1), the Hamiltonian is a polynomial of degree ν. Moreover, starting from the initial condition y_0, we want to produce a new approximation at $t = h$, say y_1, such that the Hamiltonian is conserved. Let us consider a polynomial path σ of degree $s \geq 1$. Having fixed a suitable basis $\{P_0, P_1, \ldots, P_{s-1}\}$ for Π_{s-1}, one can expand the derivative of σ as

$$\sigma'(ch) = \sum_{j=0}^{s-1} P_j(c)\gamma_j, \quad c \in [0, 1], \tag{2.4}$$

for certain set of coefficients $\{\gamma_j\}$ to be determined. In particular, we here consider an orthonormal polynomial basis, provided by the shifted and scaled Legendre polynomials on the interval $[0, 1]$.

By imposing the initial condition

$$\sigma(0) = y_0,$$

one then formally obtains

$$\sigma(ch) = y_0 + h\sum_{j=0}^{s-1} \int_0^c P_j(x)dx\gamma_j, \quad c \in [0, 1], \tag{2.5}$$

and the new approximation given by $y_1 = \sigma(h)$.

Energy conservation may be obtained by the following computation, namely

$$\begin{aligned} H(y_1) - H(y_0) &= H(\sigma(h)) - H(\sigma(0)) \\ &= \int_0^h \nabla H(\sigma(t))^T \sigma'(t)dt \\ &= h\int_0^1 \nabla H(\sigma(ch))^T \sigma'(ch)dc \\ &= h\int_0^1 \nabla H(\sigma(ch))^T \sum_{j=0}^{s-1} P_j(c)\gamma_j dc \\ &= h\sum_{j=0}^{s-1}\left[\int_0^1 \nabla H(\sigma(ch))P_j(c)dc\right]^T \gamma_j = 0, \end{aligned}$$

provided that the unknown coefficients $\{\gamma_j\}$ satisfy

$$\gamma_j = J\int_0^1 \nabla H(\sigma(ch))P_j(c)dc, \quad j = 0, 1, \ldots, s-1. \tag{2.6}$$

By setting, hereafter,

$$f(\cdot) = J\nabla H(\cdot),$$

the new approximation is then given by plugging (2.6) into (2.5):

$$\begin{aligned} y_1 = \sigma(h) &= y_0 + h\sum_{j=0}^{s-1} \int_0^1 P_j(x)dx \int_0^1 P_j(\tau)f(\sigma(\tau h))d\tau \\ &= y_0 + h\int_0^1 f(\sigma(\tau h))d\tau. \end{aligned} \tag{2.7}$$