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Four ways to compute the inverse of the complete elliptic integral of the first kind

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ABSTRACT

The complete elliptic integral of the first kind arises in many applications. This article furnishes four different ways to compute the *inverse* of the elliptic integral. One motive for this study is simply that the author needed to compute the inverse integral for an application. Another is to develop a case study comparing different options for solving transcendental equations like those in the author's book (Boyd, 2014). A third motive is to develop analytical approximations, more useful to theorists than mere numbers. A fourth motive is to provide robust "black box" software for computing this function. The first solution strategy is "polynomialization" which replaces the elliptic integral by an exponentially convergent series of Chebyshev polynomials. The transcendental equation becomes a polynomial equation which is easily solved by finding the eigenvalues of the Chebyshev companion matrix. (The numerically ill-conditioned step of converting from the Chebyshev to monomial basis is never necessary). The second approximation is a regular perturbation series, accurate where the modulus is small. The third is a power-and-exponential series that converges over the entire range parameter range, albeit only sub-exponentially in the limit of zero modulus. Lastly, Newton's iteration is promoted from a local iteration to a global method by a Never-Failing Newton's Iteration (NFNI) in the form of the exponential of the ratio of a linear function divided by another linear polynomial. A short Matlab implementation is provided, easily translatable into other languages. The Matlab/Newton code is recommended for numerical purposes. The other methods are presented because (i) all are broadly applicable strategies useful for other rootfinding and inversion problems (ii) series and substitutions are often much more useful to theorists than numerical software and (iii) the Never-Failing Newton's Iteration was discovered only after a great deal of messing about with power series, inverse power series and so on.

1. Introduction

The complete elliptic integral arises in a thousand applications, and various approximations and series for it can be found in a variety of sources such as Chapter 18 of the NIST Digital Library of Functions [1]. However, no simple form for the *inverse* is listed in any of these references. The complete elliptic integral of the first kind can be written in multiple forms.

$$K(m) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-t^2}\sqrt{1-mt^2}} \qquad (1)$$

$$= \frac{\pi}{2}\theta_3^2(0;\tau), \quad m = \frac{\theta_2^4(0;\tau)}{\theta_3^4(0;\tau)} \qquad (2)$$

$$= \text{arcsn}(\pi/2; m) \qquad (3)$$

where θ_3 is the standard Jacobian theta function and arcsn is the inverse of the elliptic sine [1]. The inverse problem is to compute $m(\lambda)$ that solves

$$K(m) - \lambda = 0, \quad \lambda \in [\pi/2, \infty], \; m \in [0, 1]. \qquad (4)$$

Unfortunately, none of these forms is much use in inverting the integral. The inverse of the Jacobian elliptic function looks promising, but this and the incomplete elliptic integral are both bivariate functions where one argument is usually a coordinate in applications and the other is the elliptic modulus m, which is typically a parameter. The elliptic sine is the inverse of the incomplete elliptic integral with respect to the *coordinate*. The inverse of $K(m)$ is the inverse with respect to the *parameter*.

Fukushima [2] catalogs some applications for $K^{-1}(m)$. Our motive was two different applications: (i) nonlinear wave theory where $K(m)$ is the period for traveling wave solutions to classical nonlinear wave equations and (ii) construction of new, more uniform grids for Chebyshev spectral methods by replacing circular functions and their inverses by elliptic functions and their inverses.

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Amore [3] shows that the singularity of the deflection angle of a light ray in the Schwarzschild metric is given by the inverse of a function derived from the elliptic integral, though not the elliptic integral itself. In other articles, Amore and Fernandez [4,5] have shown that often the best perturbation expansion for a function like $K(m)$ is a perturbation series for the inverse. Although they never treat $K^{-1}(m)$ explicitly, but only closely related functions, their work emphasizes that it is difficult to fully understand and optimally analyze a function without its inverse.

However, applications are only part of our motivation. Another goal to use this example to illustrate a quartet of rather general strategies for function inversion. The inverse elliptic integral is a good exemplar partly because it does have practical applications and partly because it has a rich history more than two centuries old. This example is also good because it is hard: the range of the argument λ of the inverse is *infinite* and the elliptic integral is logarithmically singular as $m \to 1$.

Nevertheless, three of our four methods are *global*, converging over the entire branch. The elliptic integral is a good illustration of the strengths and drawbacks of the numerical and analytic inversion methods analyzed here.

Polynomialization is an improvement on the widely-used Chebyshev-proxy rootfinder in which real roots on an interval are calculated by (i) expanding $f(x, \lambda)$ as a Chebyshev series in x on the interval and then finding the roots of this polynomial proxy by computing the eigenvalues of the Chebyshev companion matrix whose elements are trivial functions of the Chebyshev coefficients [6]. Usually the proxy must be computed anew for each λ. Here, we shall obtain a single proxy which is valid for the entire range, even though that domain in λ is unbounded.

We shall discuss each of the four inversion methods in its own section and then make some summary remarks at the end. This work is a further case study illustrating the methodologies for inversion and rootfinding discussed in the author's book [7].

2. Polynomialization of the complete elliptic integral of the first kind

It is convenient to introduce a "complementary modulus" x defined by

$$x \equiv 1 - m \tag{5}$$

because the integral is singular as $m \to 1$, which is equivalent to $x \to 0$. (In the elliptic function literature, this quantity is usually denoted m', but we have chosen x to emphasize that this parameter is the unknown in the transcendental equation solved here.) Polynomialization over the entire interval $m \in [0, 1]$ is hard because for small x

$$K(m) = K(1-x) \sim \left(-\frac{1}{2} - \frac{1}{8}x\right)\log(x) + \log(4) + \left(\frac{1}{2}\log(2) - \frac{1}{4}\right)x. \tag{6}$$

To deal with this, we use a logarithmic transformation from $x \in [0, 1]$ to $y \in [0, \infty]$:

$$x = \exp(-y). \tag{7}$$

We then apply the inverse map used to convert the rational Chebyshev functions for the semi-infinite interval into Chebyshev polynomials on $z \in [-1, 1]$ [8]:

$$y = L\frac{1+z}{1-z} \tag{8}$$

which implies

$$x = \exp\left(-L\frac{1+z}{1-z}\right). \tag{9}$$

The elliptic integral can be written without approximation as

$$K(m) = K(1-x) \sim Q(x)\,\log(x) + P(x) \tag{10}$$

where P and Q have ordinary convergent power series. Inserting the map gives

$$K(m) = K(1-x) \sim Q(x)\,\log\left(\exp\left(-L\frac{1+z}{1-z}\right)\right) + P(x) \tag{11}$$

$$= Q(x[z])\left(-L\frac{1+z}{1-z}\right) + P(x[z]) \tag{12}$$

K is thus unbounded as $z \to 1$, and therefore does not have a bounded TL series. The singular term is $-Q(x[z])L(1+z)/(1-z)$, which cannot be subtracted unless one writes a new special function routine to evaluate Q.

So, we shall instead solve $K(m = 1 - x) = \lambda$ by solving

$$K(m = 1-x)(1-z) = \lambda(1-z). \tag{13}$$

We expand $K(m = 1 - x[z])(1 - z)$ as a Chebyshev polynomial series in z up to and including degree N, $K_N(z)$, and solve the polynomial equation

$$K_N(z) - \lambda(1-z) = 0. \tag{14}$$

The mechanics of solving polynomial equations in Chebyshev form is described in the Appendix. Denote the unique zero of $K_N(z) - \lambda(1 - z)$ on $z \in [-1, 1]$ by z_*. The x value is

$$x = \exp\left(-L\frac{1+z_*}{1-z_*}\right). \tag{15}$$

The elliptic modulus $m(\lambda)$, the inverse of the elliptic integral, is

$$m = 1 - \exp\left(-L\frac{1+z_*}{1-z_*}\right). \tag{16}$$

One might expect loss of accuracy for $z \approx 1$ where $x \approx 0$ and $m \approx 1$. However, tests show little error is this limit.

Approximation of $K(1 - z)$ to near Matlab/IEEE double precision, 2×10^{-16}, can be obtained from 50 terms in the Chebyshev series and $L = 8$. The Chebyshev series in z can also be interpreted as a series of rational Chebyshev functions $TL_n(y[\lambda])$. The usual rate of convergence for rational Chebyshev series is exponential but subgeometric [9]; Fig. 1 suggests that this expectation is true. However, multiple precision was needed here to observe that the coefficients do not asymptote to a straight line (falling as $\exp(-qn)$, a geometric rate of convergence), but rather show a slight upward curvature.

It is necessary to switch to the asymptotic expansion for $|x| \le 10^{-6}$ to avoid overflow errors in 16 decimal digit precision.

$$\begin{aligned} K(1-x[z])(1-z) &\sim (1-z)\left\{\log(4) + \left(\frac{1}{2}\log(2) - \frac{1}{4}\right)x[z] \right. \\ &\quad \left. + \left(\frac{9}{32}\log(2) - \frac{21}{28}\right)x[z]^2\right\} \\ &\quad - L(1+z)\left(-\frac{1}{2} - \frac{1}{8}x[z] - \frac{9}{128}x[z]^2\right), \\ &|x| \ll 1 [m \approx 1]. \end{aligned} \tag{17}$$