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A numerical meshless method of soliton-like structures model via an optimal sampling density based kernel interpolation

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ABSTRACT

To find a numerical solution for soliton-like structures model, we propose an adaptive meshless method based on the optimal sampling density (OSD) of kernel interpolation. We first consider the relationship between the optimal nodal distribution and the error bound of kernel interpolation, and obtain the corresponding OSD. Then we introduce an OSD based kernel interpolation method to approximate a function. And a numerical two-step meshless method is finally suggested for soliton-like structures model, taking the sine-Gordon equation as an example. In each time level, the predictor process takes field nodes with the same node distribution, while the final process takes field nodes arranged adaptively according to each OSD. With only a little added computational cost, the solution accuracy can be much improved. From the numerical examples, it is shown that the proposed method is very helpful for simulating soliton-like structures model.

1. Introduction

In recent years, more and more attention has been paid for soliton-like structures model in higher dimensions [1–3]. As Argyris et al. [4] said, solitons represent essentially special wave-like solutions to nonlinear dynamic equations, and due to dispersion, these waves progress through the medium without experiencing any deformation. Furthermore, after interaction with other solitons, there appears no deformation. Soliton solutions in different analytical methods and numerical techniques for some well-known PDE, such as the Korteweg–de Vries equation, the nonlinear Schrödinger equation and the sine-Gordon equation (SGE) [4] etc., can be found in [5]. According to the representation of approximate solutions, the numerical methods for nonlinear PDEs include four main classes: the finite difference methods (FDMs), the finite element methods (FEMs), the finite volume methods (FVMs) and the spectral methods [4,6–19].

However, in the field of numerical methods for solving PDE, although the previous methods are widely used in engineering field due to their robustness and applicability, they usually require the construction and update of a mesh, and this is the main inherent disadvantage in all of these methods. Therefore, in order to overcome these difficulties, in recent years, other types of numerical techniques called meshless method have attracted researchers' attention. It can establish the system of algebraic equations for the whole problem domain without using the predefined mesh for the domain discretization. And it does not require any a priori information on the relationship between the nodes for the interpolation or approximation of unknown functions of field variables [20].

Generally, meshless methods can be divided into strong forms and weak forms. In a strong form formulation, it is assumed that the approximate unknown function should have sufficient degree of consistency, so that it is differentiable up to the order of the PDEs, and a series of meshless strong form approaches were presented in Refs. [10,21–30]. Unfortunately, a strong form of equation is difficult for practical engineering problems that are usually complex in nature. In contrast to the strong form, the weak form requires a weaker consistency on the approximate function [20]. This is used to achieve a set of algebraic equations through an integral operation using sets of background cells that may be constructed globally or locally in the problem domain. The global weak form methods are said to be not "truly" meshless method, because these methods need global background cells for numerical integration. To avoid this, the local weak form method is developed, then the numerical integrations are carried out over a local quadrature domain defined for a node. There are many researches proposed to meshless local weak form methods in Refs. [31–37].

One of the main advantages of meshless methods is that the selection of field nodes could be controlled automatically and adaptively in theory, but for most of existing methods of solving

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nonlinear PDE, the nodal distribution is preassigned [24,29,38–43]. And it is clear that an adaptive selection of nodes will be more effective and accurate than a preassigned nodal distribution to capture the soliton structure. Hence, the starting point of this paper is to build an adaptive nodal selection based meshless method for solving soliton-like structures model.

In this paper, we first consider the optimal nodal distribution of kernel interpolation for a given error. Then a predictor based method is introduced to obtain an improved kernel interpolation. Finally, a numerical meshless method is suggested for soliton-like structures model on the basis of this technique. The main characteristic of our method is that it takes different node distributions with the same number in each time step. And it selects field nodes adaptively for every time steps. This paper's outline is as follows: Section 2 presents the relationship between the sampling density and the error bound of kernel interpolation, and proposes an optimal sampling density of kernel interpolation. Section 3 obtains a stable optimal sampling density and a neighborhood importance measure of any nodes, and with the help of this optimal density, an improved kernel interpolation method is proposed to approximate a function. Then we choose $(n+1)$-dimensional SGE as an applied model, and propose an adaptive meshless method for solving it in Section 4. In Section 5, we provide two examples with a comparison of numerical error analysis. Section 6 summarizes the relevant results.

2. Kernel interpolation and its local error bound

In this paper, we will use a special kernel interpolation to approximate a function. By a kernel function, we mean a symmetric $K_\lambda(x) = K(x, \lambda)$ satisfying [44,45]

$$\int K_\lambda(x)\mathrm{d}x = 1$$

where the parameter λ is a positive number, and its selection depends on the sampling distribution of data. Moreover, let k be any positive integer, then a $2k$ orders kernel function $K_\lambda(x)$ satisfies [44,45]

$$\int_{\mathbb{R}} x^m K_\lambda(x)\mathrm{d}x = \begin{cases} 1 & m = 0 \\ 0 & 0 < m < 2k \\ C_k \lambda^{2k} & m = 2k \end{cases} \tag{1}$$

where $m = 0, 1, \ldots, 2k$ and C_k is a nonzero constant. Then we will first discuss the relationship between the sampling distribution and the interpolation error.

2.1. Sampling distribution and interpolation error

Let us start with a simple example. Suppose $f(x)$ is a function and has following expression:

$$f(x) = 4\arctan\left[\frac{1}{2}\operatorname{sech}(x)\right] - \frac{1}{3}\exp(-2x^2)\cos(10x) \tag{2}$$

where $x \in [-5, 5]$. Moreover, suppose $g_1(x)$ is an interpolating function of $f(x)$ with respect to the equally spaced nodes $\boldsymbol{\chi}_1$ with the size 21, and $g_2(x)$ is same to $g_1(x)$, but with 41 equally spaced nodes $\boldsymbol{\chi}_2$. Fig. 1 displays the graphs of $f(x)$ and its approximations. Then the corresponding absolute values of these two interpolation errors are shown in Fig. 2(a).

Besides, in order to test the performance of these interpolating functions, we use the L_2 error, L_∞ error and root-mean-square (RMS) error norms defined as

$$L_2 = \|f(x_i) - g(x_i)\|_2 = \sqrt{\frac{1}{M}\sum_{i=1}^{M} |f(x_i) - g(x_i)|^2} \tag{3}$$

$$L_\infty = \|f(x_i) - g(x_i)\|_\infty = \max_{1\leqslant i\leqslant M} |f(x_i) - g(x_i)| \tag{4}$$

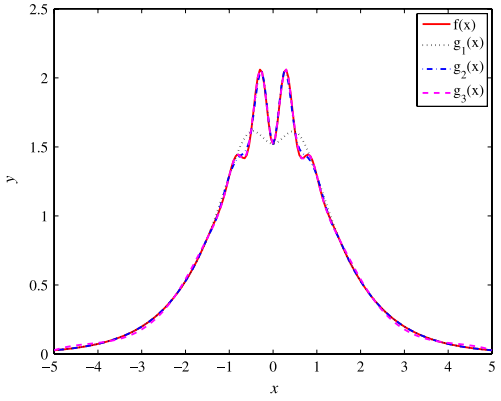

Fig. 1. The function $f(x)$ and its two approximation methods.

Table 1
A comparison of L_2, L_∞ and RMS errors of example.

Method	L_2-error	L_∞-error	RMS error
$g_1(x)$	0.1011	0.4721	0.0221
$g_2(x)$	0.0121	0.0454	0.0019
$g_3(x)$	0.0121	0.0203	0.0026

and

$$\mathrm{RMS} = \frac{1}{M}\sqrt{\sum_{i=1}^{M} |f(x_i) - g(x_i)|^2} \tag{5}$$

where M is the number of test nodes, $f(x_i)$ is the exact solution, and $g(x_i)$ is the numerical solution. The results of these errors with different interpolating functions are presented in Table 1. From these results we can notice that the estimate $g_1(x)$ in the interval $[-1, 1]$ is quite different from the original $f(x)$ and its L_∞-error is 0.4721 near $x = \pm 0.3$. If we increase the sample size from $N = 21$ to 41, that is, the interpolation function is changed from g_1 to g_2, then the L_∞ error of estimation $g_2(x)$ only decreases to 0.0454.

However, for a function approximation, a better way of the node distribution is not equally spaced, but is spaced appropriately in order to improve the accuracy of the approximation. For example, the function $f(x)$ in Eq. (2) is complex in a small region of the whole interval, and we wish to improve the accuracy of the region that the corresponding error is larger than other regions. In other words, for a given sample size N, the optimal strategy of this function is to try to move some points in the region with a small local error into somewhere with a large one. Suppose the given sample size is 21, then a special kind of nodes $\boldsymbol{\chi}_3$ (see Fig. 2(b) for details) can be developed according to an optimal sample strategy, which we will discuss in the following sections of this paper. Then we have the third interpolating function $g_3(x)$ of $f(x)$ with respect to these special nodes (see Fig. 1). From Table 1, we can find out that all kinds of interpolation errors of $g_3(x)$ are smaller than those of $g_1(x)$ and $g_2(x)$. And from Figs. 1 and 2(a), it is noted that an appropriate sampling distribution is able to improve the accuracy effectively for function approximation. And we shall give a theoretical analysis of this point in the next subsection.

2.2. Optimal sampling density of kernel interpolation

Our emphasis here is given to find out the optimal density of kernel interpolation for a given error. Suppose $K_\lambda(x)$ is a kernel function, since K_λ is an even symmetry function, it follows that:

$$\int_{-\infty}^{+\infty} K_\lambda(x - t)\mathrm{d}t = \int_{-\infty}^{+\infty} K_\lambda(t - x)\mathrm{d}t.$$

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