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Numerical simulation of second-order hyperbolic telegraph type equations with variable coefficients

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ABSTRACT

In this article, the authors proposed a numerical scheme based on Crank–Nicolson finite difference scheme and Haar wavelets to find numerical solutions of different types of second order hyperbolic telegraph equations (i.e. telegraph equation with constant coefficients, with variable coefficients, and singular telegraph equation). This work is an extension of the scheme by Jiwari (2012) for hyperbolic equations. The use of Haar basis function is made with multiresolution analysis to get the fast and accurate results on collocation points. The convergence of the proposed scheme is proved by doing its error analysis. Four test examples are considered to demonstrate the accuracy and efficiency of the scheme. The scheme is easy and very suitable for computer implementation and provides numerical solutions close to the exact solutions and available in the literature.

1. Introduction

This article is dedicated to study the numerical solutions of the second order hyperbolic telegraph type equation. Telegraph equation models an infinitesimal piece of telegraph wire as an electrical circuit. Telegraph equation describes the voltage and current in a double conductor with distance x and time t. The one-space dimension second-order linear hyperbolic telegraph equation is defined as

$$\frac{\partial^2 u(x,t)}{\partial t^2} + 2\alpha(x,t)\frac{\partial u(x,t)}{\partial t} + \beta^2 u(x,t) = A(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + g(x,t), \quad (x,t) \in [0,1] \times [0,T], \qquad (1)$$

with initial and boundary conditions

$$u(x,0) = g_1(x), \qquad \frac{\partial u}{\partial t}(x,0) = g_2(x), \qquad (2)$$

$$u(0,t) = \phi_1(t), \qquad u(1,t) = \phi_2(t), \quad t \geq 0 \qquad (3)$$

where $g,\ g_1, g_2, \phi_1, \phi_2$ are known functions and the function u is unknown. For $\alpha > 0, \beta = 0$ Eq. (1) represents a damped wave equation, and if $\alpha > \beta > 0$, it is known as telegraph equation. For the detailed study of Eq. (1), we refer [1–28] and references therein.

As we know a small telegraph wire and the long transmission line have similar characteristics. Thus, it is adequate to model an infinite small piece of telegraph wire to represent a transmission line over distance. Generally, two conductors are not perfectly insulated because of the current flow and potential difference between them. Consequently, it is a formidable task to perfectly analyse the system in order to achieve the maximum output and minimum error. To obtain maximum output in communication systems, it is necessary to determine the project power and signal failure in the system. To determine such type of failures it is essential to formulate a technique which ensures a maximum output. In terms of voltage and current, a mathematical derivation for the telegraph equation has been investigated in [1]. In signal analysis, telegraph equation is usually used for transmission and propagation of electrical signals. Telegraph equation also has applications in other fields like microwaves and radio frequency fields [3].

In recent years, much attention has been given in the literature to the development of stable methods for the numerical solutions of the second-order hyperbolic equations with constant coefficients [4,5,23]. These higher order difference methods and explicit difference methods are conditionally stable. Mohanty [7] demonstrated a new unconditionally stable technique to solve the linear one dimensional hyperbolic equation (1). In [9], authors found an approximate solution of telegraph equation by using interpolating scaling function, and in [11], Dehghan et al. combined a higher-order compact finite difference scheme of fourth order for discretizing spatial derivative of linear hyperbolic equation and collocation method for the time component

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to find the numerical solutions of one-dimensional linear hyperbolic equation. Dehghan et al. proposed some numerical methods for telegraph equation by using Chebyshev cardinal function [12], Differential quadrature method [13,21] and some other numerical methods in [19,20]. Hosseini et al. proposed a competitive numerical scheme based on Rothe's approximation for time discretization and Wavelet-Galerkin for the spatial discretization has been discussed in [14]. Xie et al. [28] *proposed* fourth-order compact difference and alternating direction implicit schemes for telegraph equation. Also in [8], authors developed a numerical scheme to solve the one-dimensional hyperbolic telegraph equation using alternating group explicit method. There is sufficient literature available on telegraph equation with constant coefficients but the literature for telegraph equation with variable coefficients is very sparse [10,22]. In this article, we design a numerical scheme using finite difference scheme and Haar wavelet method to examine telegraph equation with constant coefficients as well as with variable coefficients and singular coefficients.

As we know partial differential equations are used to describe the physical phenomena which are often difficult to solve analytically, therefore numerical methods have to be used. Numerical schemes based on Haar wavelet are efficient methods for solving such type of partial differential equations since it can track the singularity and increase the local resolution of the grid by adding higher resolutions. In smoother regions, the results can be calculated at lower resolutions. Wavelets have many excellent properties such as orthogonality, compact support, exact representation of polynomials to a certain degree and flexibility to represent functions at different levels of resolution. In numerical analysis Haar wavelet is popular due to its property of localization. Generally, writing up the operational matrices is quite tedious when we want to perform the calculation at high resolution. Haar wavelet operational matrix of integration method helps us to easily solve high order partial differential equation and nonlinear problems at high resolution. In this study, we describe how wavelets may be used in slightly different manner for the discretization of telegraph equation with Crank–Nicolson finite difference scheme. After the time discretization by Crank–Nicolson scheme, the system of ordinary differential equations is solved by projecting the solution onto the wavelet space to produce a system of algebraic equations; that can be solved by iterative methods. The approximation properties of the scaling function of the multiresolution analysis [18] provide computational efficiency and accuracy of numerical solutions of partial differential equations.

A short introduction of the Haar wavelets and its applications can be found in [15–18]. But, the essential shortcoming of Haar wavelet is: it is not continuous, that is derivatives do not exist at the point of discontinuity. Therefore it is not possible to apply the Haar wavelet directly for solution of any differential equation. There are two possibilities of ending this stand-still situation. First, the piecewise constant Haar functions can be regularized by interpolation spline but interpolation spline generates the complexity in the solution and the simplicity of Haar wavelet is lost. The other possibility is to expand all functions into Haar series. In this paper we have applied the technique of Haar series by approximating the highest derivative appearing in the differential equation. The other derivatives are obtained through integrations of this Haar series. Since the differentiation of Haar wavelet always results in impulse functions which should be avoided, the integration of Haar wavelet is preferred because it can be expanded into Haar series with the Haar coefficient matrix P on collocation points. After integrating the Haar function, we get the Haar matrices P_1 and P_2 of $2M \times 2M$ order. These matrices are then used to solve the given telegraph equation. The main idea of this composite scheme is to convert a differential equation into a system of algebraic equations, and then to discretize the algebraic equations at collocation points. The benefits of Haar wavelet transform are sparse matrix of representation, possibility of implementation of fast algorithms, more accuracy and less computation time. Error analysis of the proposed method is also carried out and it is shown that the method is convergent. The accuracy of the proposed scheme is demonstrated on four test problems. The results of numerical experiments are compared with analytical solutions and other existing methods to confirm the better accuracy of the proposed scheme.

In order to elucidate our arguments in a synchronized manner, we have summed up the paper under following sections: in Section 2, we describe the semi-discretization of telegraph equation by Crank–Nicolson scheme. Section 3 deals with Haar wavelet method for spatial discretization. We present the error analysis for the proposed scheme for solving telegraph equation in Section 4. In Section 5, we have solved four numerical examples and compared the composite Haar wavelet method with existing method discussed in [10] and analytic solutions of the differential equations via 3-dimensional and contour plots. The conclusion of the theory developed in the article is given in Section 6.

2. Semi-discretization of telegraph equation

We discretized the time derivative terms of the given equation using the Crank–Nicolson finite difference scheme. Crank–Nicolson finite difference scheme is a combination of forward Euler's method at jth and the backward Euler method at $(j+1)$th level.

Applying forward Euler method on Eq. (1), we get

$$\left(\frac{u_{j+1}-2u_j+u_{j-1}}{\Delta t^2}\right)+2\alpha\left(\frac{u_{j+1}-u_{j-1}}{2\Delta t}\right) = A(u_{xx})_j-\beta^2(u)_j+g(x,t_j),\quad 0\le j\le N-1. \tag{4}$$

Similarly backward Euler method at $(j+1)$th level in time direction on Eq. (1) gives

$$\left(\frac{u_{j+1}-2u_j+u_{j-1}}{\Delta t^2}\right)+2\alpha\left(\frac{u_{j+1}-u_{j-1}}{2\Delta t}\right) = A(u_{xx})_{j+1}-\beta^2(u)_{j+1}+g(x,t_{j+1}),\quad 0\le j\le N-1. \tag{5}$$

Adding Eqs. (4) and (5), we obtain

$$\left(\frac{u_{j+1}-2u_j+u_{j-1}}{\Delta t^2}\right)+2\alpha\left(\frac{u_{j+1}-u_{j-1}}{2\Delta t}\right) = A\left(\frac{(u_{xx})_j+(u_{xx})_{j+1}}{2}\right)-\beta^2\left(\frac{(u)_j+(u)_{j+1}}{2}\right) + \left(\frac{g(x,t_j)+g(x,t_{j+1})}{2}\right),\quad 0\le j\le N-1 \tag{6}$$

with initial and the boundary conditions

$$u_0=g_1(x),\qquad (u_0)_t=g_2(x), \tag{7}$$

$$u_{j+1}(0)=\phi_1(t_{j+1}),\qquad u_{j+1}(1)=\phi_2(t_{j+1}),\quad j=0,1,\ldots,N-1. \tag{8}$$

On simplifying Eq. (6) we get

$$2(u_{j+1}-2u_j+u_{j-1})+2\alpha\Delta t(u_{j+1}-u_{j-1}) = \Delta t^2A((u_{xx})_j+(u_{xx})_{j+1})-\Delta t^2\beta^2((u)_j+(u)_{j+1}) + \Delta t^2(g(x,t_j)+g(x,t_{j+1})),\quad 0\le j\le N-1, \tag{9}$$

where u_{j+1} is the solution of the above differential equation (6) at $(j+1)$th time step. For the solution of the system of second order linear ordinary differential equation (9) we arrange as;

$$(2+2\alpha\Delta t+\Delta t^2\beta^2)(u)_{j+1}-\Delta t^2A(u_{xx})_{j+1} = (4-\Delta t^2\beta^2)(u)_j+(-2+2\alpha\Delta t)u_{j-1}+\Delta t^2A(u_{xx})_j + \Delta t^2g(x,t_j)+\Delta t^2g(x,t_{j+1}),\quad 0\le j\le N-1, \tag{10}$$