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A derivation of energy-preserving exponentially-fitted integrators for Poisson systems

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ABSTRACT

Exponentially-fitted (EF) methods are special methods for ordinary differential equations that better compute periodic/oscillatory solutions. Such solutions often appear in Hamiltonian systems, and in view of this, symplectic or energy-preserving variants of EF methods have been intensively studied recently. In these studies, the symplectic variants have been further applied to Poisson systems, while such a challenge has not ever been done for the energy-preserving variants. In this paper, we propose an energy-preserving EF method for Poisson systems, with special emphasis on the second- and fourth-order schemes.

1. Introduction

In this paper,[1] we consider the numerical integration of Poisson systems of the form

$$\dot{y} = \Lambda(y)\nabla H(y), \qquad y(t_0) = y_0, \tag{1}$$

where $y \in \mathbb{R}^n$, $\Lambda(y)$ is a skew-symmetric matrix and the dot on y stands for the differentiation with respect to time. It is well known that the energy (Hamiltonian) $H(y)$ is constant along the solution, since

$$\dot{H}(y(t)) = \nabla H(y(t))^\top \dot{y}(t) = \nabla H(y(t))^\top \Lambda(y(t))\nabla H(y(t)) = 0,$$

and Poisson systems often have periodic or oscillatory solutions. With these considerations, the aim of this paper is to construct energy-preserving integrators specially tuned for periodic or oscillatory solutions.

There have been a lot of studies on energy-preserving methods, as well as symplectic methods, for Hamiltonian systems in which the matrix Λ is independent of y (see [2–7] and references therein). While symplectic methods have been well studied and often produce nice numerical solutions, there are several advantages of adopting energy-preserving methods. For example, it is easier to adapt stepsize control for energy-preserving methods than for symplectic methods, and for chaotic dynamics symplectic methods are occasionally unstable. The average vector field (AVF) method [7][2]

$$y_1 = y_0 + h\Lambda \int_0^1 \nabla H(y_0 + \tau(y_1 - y_0))\,\mathrm{d}\tau \tag{2}$$

is a unified way of constructing energy-preserving integrators for Hamiltonian systems (see also [4,6] for the discrete gradient method). The AVF method is of order two. Hairer extended the method to higher order by introducing continuous stage Runge–Kutta (CSRK) methods [3]. However, Poisson systems (1) require an additional technique, due to the dependence of the matrix Λ on $y(t)$: $\Lambda(y)$ and $\nabla H(y)$ should be discretised in a different manner. The simplest second-order discretisation is given by

$$y_1 = y_0 + h\Lambda\left(\frac{y_0 + y_1}{2}\right)\int_0^1 \nabla H(y_0 + \tau(y_1 - y_0))\,\mathrm{d}\tau, \tag{3}$$

which can also be seen as an extension of the AVF method (2). This example suggests the necessity of introducing an idea of partitioned methods. Cohen–Hairer [8] (see also Brugnano et al. [9]) succeeded in constructing arbitrary high-order energy-preserving schemes for Poisson systems (1).

On the other hand, for ordinary differential equations with periodic or oscillatory solutions, there have been a lot of research

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[1] This work is a full version of our recent report [1] with detailed discussions. In particular, Theorems 3 and 4 are reported in [1].

[2] In this paper, every numerical integrator is formulated as a one-step method $y_0 \mapsto y_1 (\approx y(t_0 + h))$ with a constant stepsize h, since we are concerned with only autonomous problems.

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activities. For example, trigonometric methods for second-order ODEs and exponentially-fitted methods for first-order ODEs have been studied in the last few decades. Among them, we focus on exponentially-fitted (EF) methods (see [10–25] and references therein). The idea of EF methods is to construct numerical one-step methods which exactly solve problems whose solution belongs to the linear space spanned by

$$\mathcal{F} = \{\exp(\lambda_1 t), \ldots, \exp(\lambda_r t)\}, \quad \lambda_i \in \mathbb{C}.$$

Note that the function of this $\mathcal{F}$ are $\mathbb{R} \to \mathbb{C}$ whereas the solutions of Hamiltonian or Poisson systems are vector functions. Hence the fitting space must have n-dimensional functions.

Recently, symplectic EF methods have been developed for Hamiltonian systems (see, e.g., [26–34]), so that the combined methods are more effective than standard symplectic methods and standard EF methods. Moreover, the methods can be (at least formally) applied to Poisson systems, and in some papers above this has been done. For both systems, numerical experiments showed the effectiveness of the methods, though strictly speaking, for Poisson systems, there has been no rigorous, theoretical discussions as far as the author understands. From the perspective of geometric numerical integration, these works motivate us to consider energy-preserving EF counterparts. The present author proposed an energy-preserving EF method for Hamiltonian systems by introducing an algebraic condition of energy-preservation in terms of CSRK methods [35]. Although this method can be simply applied to Poisson systems like as in the symplectic versions, in this paper we do not take this approach since such a formal application should obviously destroy the energy-preservation of Poisson systems, and thus such an approach does not make good sense. Instead, in this paper, we aim at a stronger result: namely, we give a new, rigorous framework for constructing EF schemes which exactly inherit the energy-preservation property of Poisson systems. When the new method is applied to Hamiltonian systems, it reduces to the existing energy-preserving EF method. The new methods have the advantages of both energy-preserving methods and EF methods.

The construction of the new method is based on the so called partitioned CSRK (PCSRK) methods. We show characterisations of energy-preservation and symmetry properties in terms of PCSRK methods, and based on them we give the new framework. There, the biggest difficulty is that a PCSRK method contains more parameters than a CSRK method, and all parameters have to be determined in terms of elementary functions. We illustrate how this can be done taking the derivation of second- and fourth-order schemes as our working examples.

This paper is organised as follows. In Section 2.1, we give energy-preservation and symmetry conditions for Poisson systems in terms of partitioned CSRK (PCSRK) methods. In Section 2.2, we introduce the concept of EF methods by illustrating the derivation of energy-preserving EF CSRK methods for Hamilton systems. In Section 3, we derive second- and fourth-order energy-preserving EF schemes. Section 4 is devoted to numerical experiments, where we consider the Euler equation. In Section 5, we conclude this paper.

2. Preliminaries

2.1. Characterisations of energy-preserving methods

We first summarise energy-preservation and symmetry characterisations for Hamiltonian systems (i.e., Λ is a constant skew-symmetric matrix) in terms of CSRK methods [35]. After that we give characterisations of PCSRK methods being energy-preserving and symmetric for Poisson systems.

We consider an s-degree CSRK method defined by

$$Y_\tau = y_0 + h\Lambda \int_0^1 A_{\tau,\sigma} \nabla H(Y_\sigma)\, \mathrm{d}\sigma, \tag{4}$$

$$y_1 = Y_1, \tag{5}$$

where Y_τ ($\tau \in [0, 1]$) is a polynomial in τ of degree s and satisfies $Y_0 = y_0$, and $A_{\tau,\sigma}$ with the assumption $A_{0,\sigma} = 0$ is a polynomial in τ and σ.

Theorem 1 (*[35]*). *A CSRK method solving Hamiltonian systems is energy-preserving if* $\frac{\partial}{\partial \tau} A_{\tau,\sigma}$ *is symmetric.*[3]

In fact, coefficient polynomials $A_{\tau,\sigma}$ derived in [3] satisfy this condition. This theorem also indicates that $A_{\tau,\sigma}$ is polynomial of degree s and $s-1$ with respect to τ and σ, respectively. Symmetry condition is also written in terms of the coefficient polynomial.

Theorem 2 (*[3]*). *A CSRK method is symmetric if*

$$A_{1-\tau,1-\sigma} + A_{\tau,\sigma} = A_{1,\sigma}.$$

Next, let us consider Poisson systems (the following characterisation was already pointed out in our recent report [35], but we here add a proof). We consider an s-degree PCSRK method defined by

$$Y_\tau = y_0 + h \sum_{j=1}^{s} \int_0^1 A_{i,\tau,j,\sigma} \Lambda(W_j) \nabla H(Y_\sigma)\, \mathrm{d}\sigma, \tag{6}$$

$$W_i = z_0 + h \sum_{j=1}^{s} \int_0^1 \widehat{A}_{i,\tau,j,\sigma} \Lambda(W_j) \nabla H(Y_\sigma)\, \mathrm{d}\sigma \quad (i = 1, \ldots, s), \tag{7}$$

$$y_1 = y_0 + h \sum_{i=1}^{s} \int_0^1 B_{i,\tau} \Lambda(W_i) \nabla H(Y_\tau)\, \mathrm{d}\tau, \tag{8}$$

$$z_1 = z_0 + h \sum_{i=1}^{s} \int_0^1 \widehat{B}_{i,\tau} \Lambda(W_i) \nabla H(Y_\tau)\, \mathrm{d}\tau, \tag{9}$$

with $y_0 = w_0$, where

- Y_τ is a polynomial in τ of degree s and satisfies $Y_0 = y_0$,
- $A_{i,\tau,j,\sigma}$ is a polynomial in τ and σ with $A_{i,0,j,\sigma} = 0$,
- $0 \le c_1 < \cdots < c_s \le 1$,
- $\widehat{A}_{i,\tau,j,\sigma} = A_{c_i,j,\sigma}$ (the notation of $A_{c_i,j,\sigma}$ is defined below),
- $B_{j,\sigma} = \widehat{B}_{j,\sigma} = A_{1,j,\sigma}$.

The notation $A_{i,\tau,j,\sigma}$ depends on τ, $\sigma \in [0, 1]$, $j = 1, \ldots, s$ and i. In reality, it does not depend on i, but we leave it as it is because it becomes useful when considering order conditions. In other places, we can simply understand this as $A_{i,\tau,j,\sigma} = A_{\tau,j,\sigma}$.

It is clear that $y_1 = w_1$ and (7) is equivalent to $Z_i = W_{c_i}$. For example, when $s = 1, A_{\tau,1,\sigma} = \tau$ and $c_1 = 1/2$, the PCSRK method reduces to (3). As mentioned in [8], these methods are consistent with the partitioned system of differential equations

$$\dot{y} = \Lambda(w)\nabla H(y), \qquad y(t_0) = y_0,$$
$$\dot{w} = \Lambda(w)\nabla H(y), \qquad z(t_0) = z_0,$$

whose solutions satisfy $y(t) = w(t)$ if $y_0 = w_0$.

Theorem 3. *A PCSRK method solving Poisson systems is energy-preserving if* $\frac{\partial}{\partial \tau} A_{i,\tau,j,\sigma}$ *is symmetric for all* $j = 1, \ldots, s$.

[3] In this paper, a polynomial $f(\tau, \sigma)$ is said to be symmetric if $f(\tau, \sigma) = f(\sigma, \tau)$.