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Analysis of cornea curvature using radial basis functions – Part I: Methodology

G.W. Griffiths [a,*], Ł. Płociniczak [b], W.E. Schiesser [c]

[a] City University, Northampton Square, London EC1V 0HB, UK
[b] Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland
[c] Lehigh University, D118 Iacocca, 111 Research Drive, Bethlehem, PA 18015, USA

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ABSTRACT

We discuss the solution of *cornea curvature* using a *meshless method* based on *radial basis functions* (RBFs). A full two-dimensional nonlinear thin membrane *partial differential equation* (PDE) model is introduced and solved using the *multiquadratic* (MQ) and *inverse multiquadratic* (IMQ) RBFs. This new approach does not rely on radial symmetry or other simplifying assumptions in respect of the cornea shape. It also provides an alternative to corneal topography modeling methods requiring accurate material parameter values, such as Young's modulus and Poisson ratio, that may not be available. The results show good agreement with published corneal data and allow back calculations for estimating certain physical properties of the cornea, such as *tension* and *elasticity coefficient.* All calculations and generation of graphics were performed using the R language programming environment [34] and RStudio, the integrated development environment (IDE) for R [36], both of which are open source and free to download.

Part II [48] of this paper demonstrates how the method has been used to provide a very accurate fit to a corneal measured data set.

1. Introduction

The cornea is one of the most important parts of the human (and animal) eye – Fig. 1. It is an important integral part of the remarkable process of vision. In order to see, our eyes have to transform the incoming light signal (photons) into a series of electrical signals which can then be decoded by the brain into an image that we perceive. This is a truly complex process which can successfully take place only when all the organs are normal and healthy: the eye is a place where biomechanics meets optics. As the incident light beam enters the eye it is being refracted by the cornea, then passes through the lens which focuses it onto the rear of the eye – the retina. It then starts a series of processes producing electrical impulses which travel to the brain via the optic nerve. The shape of the cornea is controlled by the *intraocular pressure* (IOP) of the anterior chamber. This usually varies between 10 and 21 mmHG, and is generated by the continuous production of and outflow of aqueous humour in the eye.

The reader can find a thorough exposition of the eye anatomy in [21] while a more physical description of the process of vision can be found, for example, in [2]. In this paper we concentrate on mathematically modeling corneal topography.

The geometrical and mechanical properties of the cornea are very important since this organ plays a crucial role in vision. First, from the structural point of view, it has to be very durable and strong in order to withstand various external risks and dangers such as damage due to contact with foreign substances (for a review of corneal biomechanics see the works [7,23]). Second, the important optical properties of the cornea make it very transparent in order for a light beam to refract undisturbed (a molecular treatment can be found for example in [17]). Lastly, the refraction has to be done in a precise way to allow the photons to arrive exactly on the retina – not in front of it nor behind (which leads to myopia and hyperopia) (for an exposition see [8,10]). Ophthalmologists measure corneal topography in order to diagnose many vision disorders such as the aforementioned myopia and hyperopia, and also keratoconus and astigmatism (see [6,24]). Understanding the foundations of the shape and curvature of the cornea is thus indispensable in optometry, and this can be achieved through the use of mathematics.

A number of different mathematical models of the cornea have been developed throughout the last century. Probably the first scientist that conducted engineering research into the cornea was H. von Helmholtz[1] during the 19th century [18]. He proposed conic sectional surfaces of revolution to model the corneal topography

* Corresponding author.
E-mail addresses: graham@griffiths1.com (G.W. Griffiths), lukasz.plociniczak@pwr.edu.pl (Ł. Płociniczak), wes1@Lehigh.edu (W.E. Schiesser).

[1] Hermann Ludwig Ferdinand von Helmholtz (1821–1894) was a German physician and physicist.

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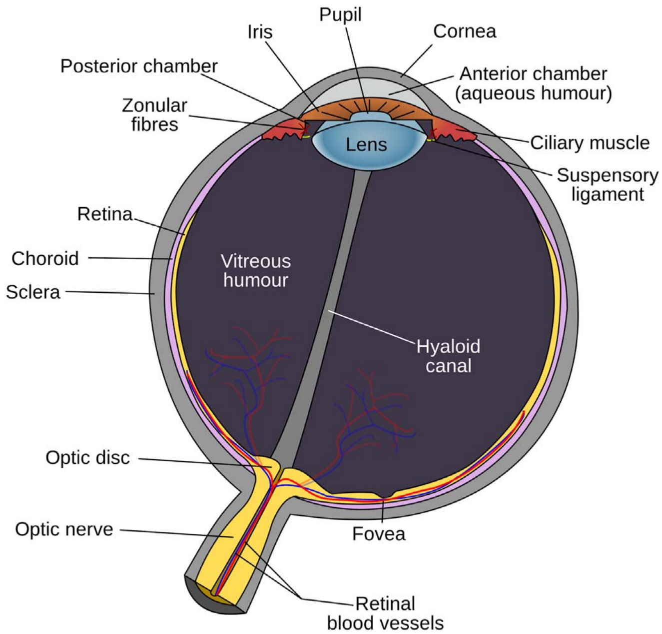

Fig. 1. Schematic diagram of human eye. Image by R.H. Castilhos, reproduced with permission.

(ellipsoids and paraboloids). Up to this day, these models are very accurate and are used widely (see e.g. [5,26]). As they were intended, conical section models are mostly empirical and are based on the fact that in the first approximation, the cornea resembles a surface of revolution. These simple models have also been used to build more accurate and realistic, structural mechanical descriptions of the material properties of the cornea (see [1]). Several approaches to that problem were undertaken, for instance based on a shell theory [44], sandwich theory [28], full equations of linear elasticity [14] or a purely numerical *finite element model* (FEM) [30,1,22,35]. Moreover, a very frequent approach to modeling corneal features (height data and optical wavefronts) is based on an expansion in a series of orthogonal Zernike polynomials which are suited to describe aberrations in optics [19,43] Also, we would like to mention a very recent approach which uses the Brillouin optical microscopy to infer into the three–dimensional structure of the cornea and eye as a whole [39,40]. All of these methods are used to test the underlying theory and verify various experiments conducted to learn about the biomechanical properties of the eye. Since the cornea has strong nonlinear characteristics, this is an important and non-trivial task.

In this paper we extend the analysis of our previous work on a model of intermediate complexity [33], lying between simple conical sectional models and very complex structural mechanical ones. More specifically, we model the cornea in a limit of vanishing thickness with an elastic nonlinear membrane shaped by the intraocular pressure. We relax some of our previously used simplifying assumptions, and provide a numerical analysis and its implementation using a meshless method with radial basis functions.

2. Mathematical model

2.1. Corneal topology

Our previous paper described a *false transient* analysis and solution to cornea curvature, as modeled by a 1D *thin plate* mathematical model [33]. This paper extends the 1D results by applying the following thin-plate 2D *boundary value* PDE curvature model to the cornea, with the calculations being performed over the spatial domain Ω, assumed elliptical, and boundary $\partial\Omega$,

$$T\nabla\cdot\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right)-kh+\frac{P}{\sqrt{1+|\nabla h|^2}}=0,\quad h=h(x),\quad x\in\mathbb{R}^2,$$
$$\text{BCs: }\partial\Omega=0,\quad \nabla h|_{x=0}=0, \tag{1}$$

where BCs indicate boundary conditions, $h(x)$ represents the height of the cornea over its surface. The maximum height $h_{\max}$, is the so-called ocular *sagitta* or *sagittal height/depth*, from geometric theory. The constants represent physical characteristics of the cornea, assumed to be isotropic, and are defined as: *tension* T [N/m], *elasticity coefficient* k [N/m^3] and *intraocular pressure* P [N/m^2].

We model the spatial domain as an ellipse with semi-major and semi-minor axes r_a and r_b. Letting $R=r_a$ represent a typical linear length associated with the cornea, and normalizing, we obtain

$$\nabla\cdot\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right)-ah+\frac{b}{\sqrt{1+|\nabla h|^2}}=0,\quad h=h(x),\quad x\in\mathbb{R}^2,$$
$$\text{BCs: }\partial\Omega=\Omega|_{r_{ab}=1}=0,\quad r_{ab}=\left(\frac{x_1^2}{r_a^2}+\frac{x_2^2}{r_b^2}\right),\quad \nabla h|_{x=0}=0. \tag{2}$$

where $a{:=}kR^2/T$ and $b{:=}PR/T$ in order for h to be nondimensional (scaled by R).

When Eq. (2) is expanded in Cartesian coordinates (x,y), see Appendix A, we obtain the following equivalent form:

$$\frac{-2h_xh_{yx}h_y+h_{xx}+h_{xx}h_y^2+h_{yy}+h_{yy}h_x^2}{\left(1+h_x^2+h_y^2\right)^{3/2}}-ah$$
$$+\frac{b}{\sqrt{1+h_x^2+h_y^2}}=0,\quad h=h(x,y), \tag{3}$$

where we have used subscript notation for representing partial derivatives, i.e. $h_x=\partial h/\partial x$, etc. Looking at the front of the cornea, the horizontal axis is represented by x and the vertical axis by y.

We model the cornea on a two-dimensional Cartesian domain defined by the boundary between the cornea and the remainder of the eye – refer to Fig. 1. This boundary is often referred to as the *corneal–limbal ring* (CLR), the dark ring around the *iris*, and we adopt this terminology.[2]

We take the CLR to be a planar ellipse in shape, having semi-major and semi-minor axes r_a and r_b respectively, with small eccentricity [29]. Again we elect to apply the *method of false transients* and this requires modifying the above boundary value, *time-independent* PDE to a *time-dependent* PDE by adding a time derivative term to the right hand side. Eq. (2) is therefore modified to become the following *initial value* (Cauchy) problem:

$$\frac{\partial h}{\partial t}=\nabla\cdot\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right)-ah+\frac{b}{\sqrt{1+|\nabla h|^2}},$$
$$h=h(x,t),\quad t\geq 0,\quad x\in\mathbb{R}^2,$$
$$\text{BCs: }\partial\Omega=\Omega|_{r_{ab}=1}=0,\quad r_{ab}=\left(\frac{x_1^2}{r_a^2}+\frac{x_2^2}{r_b^2}\right),$$
$$\nabla h|_{x=0}=0,\quad \text{IC: }h(x,0)=h_0(x), \tag{4}$$

[2] This may not be strictly accurate for elderly subjects when the CLR may become difficult to identify precisely.