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Heat treatment modelling using strongly continuous semigroups

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ABSTRACT

In this paper, mathematical simulation of bioheat transfer phenomenon within the living tissue is studied using the thermal wave model. Three different sources that have therapeutic applications in laser surgery, cornea laser heating and cancer hyperthermia are used. Spatial and transient heating source, on the skin surface and inside biological body, are considered by using step heating, sinusoidal and constant heating. Mathematical simulations describe a non-Fourier process. Exact solution for the corresponding non-Fourier bioheat transfer model that has time lag in its heat flux is proposed using strongly continuous semigroup theory in conjunction with variational methods. The abstract differential equation, infinitesimal generator and corresponding strongly continuous semigroup are proposed. It is proved that related semigroup is a contraction semigroup and is exponentially stable. Mathematical simulations are done for skin burning and thermal therapy in 10 different models and the related solutions are depicted. Unlike numerical solutions, which suffer from uncertain physical results, proposed analytical solutions do not have unwanted numerical oscillations.

1. Introduction

In recent years, temperature predictions for living tissues have great attraction due to its significance in basic and clinical sciences. The thermal wave model of bioheat transfer in living tissues reduces to one differential equation (for example see: [1,2]). Although this phenomenon is three-dimensional (3D), for many researchers the depth effects of the heat propagation is a matter of importance. Thus, without loss of generality, one can assume the heat energy propagation along the direction perpendicular to the skin surface. Liu et al. [3] introduced the thermal wave model to investigate physical mechanisms and the behaviours in living tissues. Traditionally, finite difference, boundary element and finite element methods are used [4–8] to model and solve different bioheat transfer problems numerically. However, if both closed form analytical and numerical solutions exist, the analytical one is preferred, since it does not depend on the dimensions of the problem. Moreover, in contrast to numerical solutions the analytical solutions are not depended on the previous grids in the domain. But for most of the numerical solutions of thermal wave model there are oscillations that are hard to dampen [9–11] and these solutions are physically doubtful results for the temperature prediction at the pulse surface heat flux. It is for these reasons that we aimed in this paper to present several closed form analytical solutions to the thermal wave model of bioheat transfer under different types of boundary conditions. Up to now, Haji-Sheikh et al. described a method of solution of the thermal wave equation in finite bodies using Green's functions [12]. Ahmadikia et al. [13–15] derived the solution of the hyperbolic heat equation using the Laplace transform. This paper deals with the closed analytical form solution of the thermal wave equation based on strongly continuous semigroup (C_0-semigroup) theory. Proposed general solutions in this paper work for different kinds of source terms. Therefore, they are flexible in calculating the temperature distribution for various practical problems in skin burning and hyperthermia. Furthermore, the C_0-semigroup solution is capable to dealing with the transient or space-dependent boundary conditions. In the recent years, Malek et al. [16,17] have performed several closed forms of analytical solutions for Pennes' and dual phase lag equation using C_0-semigroup theory. By semigroup theory, an infinitesimal generator is used to establish the abstract differential equation related to the original thermal wave equation. The main results consist of computing eigenvalues and proof for the well-posedness of related operator. It is proved that an infinitesimal generator is Riesz spectral operator and the corresponding system is exponentially stable.

The paper is organized as follows: in Section 2 mathematics formulation for thermal wave equation under generalized boundary conditions are described. In Section 3 semigroup formulation and closed analytical form solution are achieved. The mathematical simulation results based on semigroup theories derived for four

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different boundary conditions in Section 4. Concluding remarks are given in Section 5.

2. Mathematical formulation

In order to study thermal behaviour in living tissues, several models describing bioheat transfer have been developed [18]. Most of the thermal medical practices are based on the well-known Pennes' equation. It is based on an infinitely fast propagation of temperature disturbance (Fourier's law). In contrast with Fourier's constitutive law, a modified flux model based on the finite speed of propagation of heat in the living body was suggested [19,20]. The assumption of finite speed wave propagation does not obey in the classical Fourier's law. The thermal wave theory based on non-Fourier's law can be derived from

$$q(\vec{r}, t+\tau_q) = -k\nabla T(\vec{r}, t) \tag{1}$$

where t is the time, k is the thermal conductivity, q is the heat flux, ∇T is the temperature gradient, $\vec{r} = (x, y, z)$ is three-dimensional position vector in which z stands for the tissue depth. The relaxation time is $\tau_q = \alpha / C^2$ where α is the thermal diffusivity and C stands for the heat propagation velocity. In contrast with common metals the relaxation time for heating processes is much longer than 10^{-8} s [10]. For the processed meat Mitra et al. [21] took $\tau_q = 15$ s, while in Refs. [22,23] it is shown that τ_q is in the range of 20–30 s for biological bodies. Pennes' bioheat transfer equation is [24]

$$-\nabla q + w_b c_b (T_b - T) + Q_m + Q_r = \rho c \frac{\partial T}{\partial t}. \tag{2}$$

Here, ρ, c and T denote density, specific heat and temperature of tissue, respectively. c_b and w_b are the specific heat and perfusion rate of blood respectively. Q_m is the metabolic heat generation and Q_r is the heat source for spatial heating. T_b is the arterial temperature and was regarded as a constant. Applying the first-order Taylor's series expansion to Eq. (1) leads to

$$q(\vec{r}, t) + \tau_q \frac{\partial q(\vec{r}, t)}{\partial t} = -k\nabla T(\vec{r}, t). \tag{3}$$

By taking the gradient with respect to r on both sides of (3) and substitute in (2) one can write

$$\nabla \cdot (k\nabla T) + w_b c_b (T_b - T) + Q_m + Q_r + \tau_q \left(-w_b c_b \frac{\partial T}{\partial t} + \frac{\partial Q_m}{\partial t} + \frac{\partial Q_r}{\partial t} \right) = \rho c \left(\tau_q \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} \right). \tag{4}$$

This hyperbolic heat transfer equation is more complicated than Pennes' equation (2) since for $\tau_q = 0$, Eq. (4) reduces to Eq. (2). For constant thermal parameters k, Q_m the z-direction form of Eq. (4) is

$$k\frac{\partial^2 T}{\partial z^2} + w_b c_b (T_b - T) + Q_m + Q_r + \tau_q \left(-w_b c_b \frac{\partial T}{\partial t} + \frac{\partial Q_r}{\partial t} \right) = \rho c \left(\tau_q \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} \right). \tag{5}$$

In order to solve this equation we take

$$T(z, 0) = T_0(z) \quad \text{and} \quad \left.\frac{\partial T(z, t)}{\partial t}\right|_{t=0} = 0, \tag{6}$$

in which $T_0(z)$ for every $z \in [0, l]$ is the solution of the following ordinary differential equation [25]:

$$\begin{aligned} &k\frac{d^2 T_0(z)}{dz^2} + w_b c_b (T_b - T_0(z)) + Q_m = 0 \\ &-k\frac{dT_0(z)}{dz} = h_0 (T_f - T_0(z)), \quad z = 0, \\ &T_0(z) = T_c, \quad z = l, \end{aligned} \tag{7}$$

where $T(z, 0) = T_0(z)$ is the initial heat value at starting, T_f is the ambient temperature, h_0 is the coefficient corresponding to the ambient and skin surface temperature and T_c is the body core temperature at $z = l$. The closed form analytical solution for Eq. (7) can be written as

$$T_0(z) = T_b + \frac{Q_m}{w_b c_b} + \frac{\left(T_c - T_b - \frac{Q_m}{w_b c_b} \right) \left[\sqrt{A} \cosh(\sqrt{A} z) + \frac{h_0}{k} \sinh(\sqrt{A} z) \right]}{\sqrt{A} \cosh(\sqrt{A} l) + \frac{h_0}{k} \sinh(\sqrt{A} l)} + \frac{\frac{h_0}{k} \left(T_f - T_b - \frac{Q_m}{w_b c_b} \right) \sinh(\sqrt{A}(l - z))}{\sqrt{A} \cosh(\sqrt{A} l) + \frac{h_0}{k} \sinh(\sqrt{A} l)}, \tag{8}$$

where $A = w_b c_b / k$.

In practice, different types of boundary conditions (BCs) may be considered with Eq. (5) and initial conditions (6): (a) a transient surface heat flux, (b) a surface and body core heating, (c) a cooling medium on tissue surface, (d) a surface heating with body core heat flux, which can be generalized as

$$-k\left.\frac{\partial T(z, t)}{\partial z}\right|_{z=0} = f_0(t), \quad T(l, t) = T_c, \tag{9a}$$

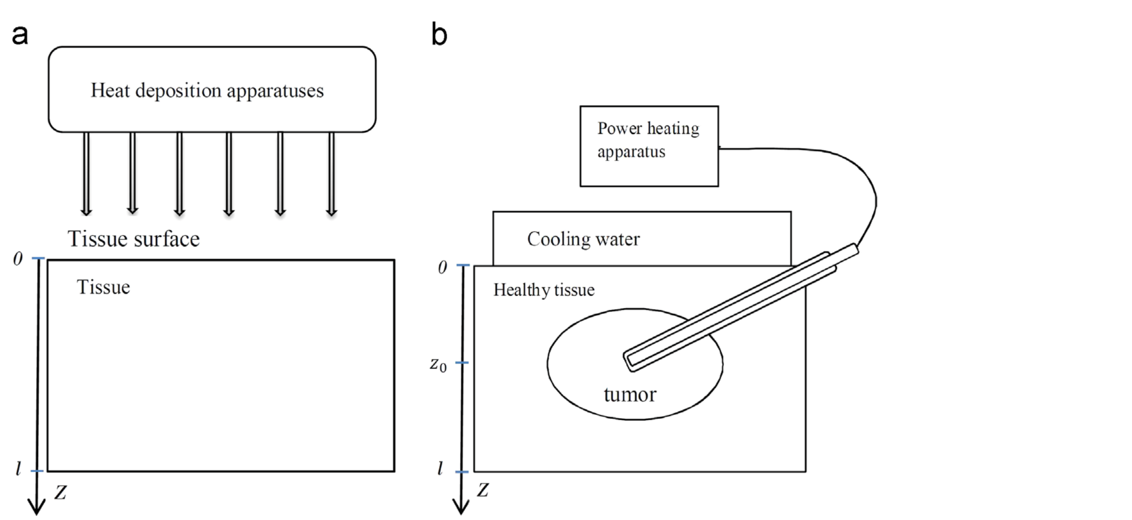

Fig. 1. Schematic of a tissue model: (a) skin burns by the tissue surface heating. (b) Hyperthermia by the point heating power, heat is deposited through inserting a heating probe in the center of tumor site ($z = z_0$).