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Discontinuous Galerkin finite element method for solving population density functions of cortical pyramidal and thalamic neuronal populations

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ABSTRACT

Compared with the Monte Carlo method, the population density method is efficient for modeling collective dynamics of neuronal populations in human brain. In this method, a population density function describes the probabilistic distribution of states of all neurons in the population and it is governed by a hyperbolic partial differential equation. In the past, the problem was mainly solved by using the finite difference method. In a previous study, a continuous Galerkin finite element method was found better than the finite difference method for solving the hyperbolic partial differential equation; however, the population density function often has discontinuity and both methods suffer from a numerical stability problem. The goal of this study is to improve the numerical stability of the solution using discontinuous Galerkin finite element method. To test the performance of the new approach, interaction of a population of cortical pyramidal neurons and a population of thalamic neurons was simulated. The numerical results showed good agreement between results of discontinuous Galerkin finite element and Monte Carlo methods. The convergence and accuracy of the solutions are excellent. The numerical stability problem could be resolved using the discontinuous Galerkin finite element method which has total-variation-diminishing property. The efficient approach will be employed to simulate the electroencephalogram or dynamics of thalamocortical network which involves three populations, namely, thalamic reticular neurons, thalamocortical neurons and cortical pyramidal neurons.

1. Introduction

To simulate brain activities, it is necessary to develop an efficient and effective modeling method because information in the brain is processed in large-scale neuronal networks across many functionally specialized areas, each of which has a myriad number of interconnected neurons [1]. The population density method is developed for this purpose, in which neurons in a large scale neuronal network are grouped into neuronal populations based on the types of neurons' dynamics, and each neuronal population has its own population density function (PDF) representing the distribution of states of all neurons in the population [2,3]. Such a method can serve as a computational tool to efficiently evaluate the population dynamics of a neuronal population consisted of homogeneous yet slightly different neurons by using statistical information of their state variables. In addition, it can also serve as a theoretical method to explore the dynamics of a large-scale neuronal system consisted of heterogeneous neurons through the interaction arising from subpopulations consisted of homogeneous neurons [4,5]. In contrast to neural mass models [6–8], the relationship between the brain activity and the specific dynamics of neurons can be built based on the model of individual neurons in PDF. It is believed that the population density method is applicable for investigating generation mechanisms of many brain activities. To trace the population dynamics, the hyperbolic partial differential equation (PDE), that is, the governing equation of the PDF, has to be solved to obtain the time evolution of the PDF.

Up to now, two common types of neuronal models are used in the population density approach to model large-scale neuronal networks in the brain. One of them is the leaky integrate-and-fire (LIF) model that can simulate the tonic firing mode of cortical pyramidal neurons [9]. By neglecting the induced dynamics of presynaptic inputs, the

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governing equation of LIF model can be simplified to a one-dimensional hyperbolic PDE [4,5]. The other is the leaky integrate-and-fire-or-burst (LIFB) model that can simulate the transition between the tonic and the burst firing modes manifested by thalamic neurons [10]. Similarly, by neglecting the induced dynamics of presynaptic inputs, the governing equation of LIFB model can be simplified to a two-dimensional hyperbolic PDE [11]. Unfortunately, even for the one-dimensional governing equation, solutions of the above models are intractable by analytic methods due to its high nonlinearity.

Numerical schemes are usually used to solve the governing equation, especially the finite difference methods [4,5] or finite volume methods [11]. The finite element method (FEM) had never been utilized until we proposed the continuous Galerkin FEM in our previous work [12]. It was concluded that the continuous Galerkin FEM has higher computational efficiency and lower discretization error when compared with the finite difference method. In addition, the geometrical flexibility of FEM is critical to deal with sophisticated neuron models that usually have complicated geometries within the computational domain. However, it lacks the property of so-called total variation diminishing so that it becomes unstable when the solution of the governing equation exhibits sharp variation within a small spatial interval or even discontinuity [13,14]. For the LIFB model, the solution could be discontinuous along a certain dimension if one state variable lost its dynamical behavior. As a result, an alternative numerical scheme must include the total-variation-diminishing property.

In this work, the discontinuous Galerkin FEM is used because it possesses the total-variation-diminishing property and, meanwhile, reserves the advantages of the continuous Galerkin FEM [15]. In the discontinuous Galerkin FEM, the continuity of solution values at inter-boundaries between meshes is released, and the total-variation-diminishing property is maintained by using a slope limiter. Additionally, the discontinuous Galerkin FEM allows parallel computing to increase computational efficiency [16]. So, in terms of the characteristics of lower discretization error, higher geometry flexibility, superior computational efficiency and parallel computing, it is believed that the discontinuous Galerkin FEM is a promising approach in solving the governing equation of the PDF. The goal of this work is to formulate the discontinuous Galerkin FEM and apply it to solve two populations in a neuronal network, namely, the cortical pyramidal and the thalamic neuron populations. The results will be compared with those obtained from the Monte Carlo method. Convergence and accuracy of solution and numerical stability of the discontinuous Galerkin FEM will be discussed.

The structure of this article is as follows: In section 2, two governing equations derived from the LIF and LIFB models are introduced. In section 3, the implemented algorithms of the discontinuous Galerkin FEM are presented for solving governing equations of LIF and LIFB models. In section 4, dynamical behaviors of neuronal populations are simulated to demonstrate the convergence and accuracy of the discontinuous Galerkin FEM. Discussion and conclusions are, respectively, presented in section 5 and section 6.

2. Governing equations

The first step of simulating neuronal population dynamics by means of the population density method is to derive the neuronal model that describes how action potentials (spikes) are generated in a neuron. After deriving the neuronal models, the governing equations based on these neuronal models are described, and the implemented algorithm of the discontinuous Galerkin FEM will be presented in the next section.

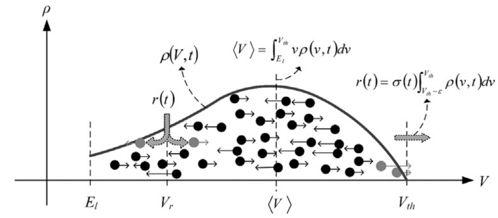

Fig. 1. Systematic illustration of population density method based on the LIF model. In this method, alteration of neurons' states caused by leaky and post-synaptic currents are imagined as moving particles in a virtue state space which becomes one-dimensional and is enclosed by two boundary points, E_l and V_{th}, respectively, if neglecting the dynamics of post-synaptic currents[4,5]. The population density function, $\rho(V,t)$, in which the membrane potential, V, is the unique state variable, represents the distribution of neurons in that space. The average number of neurons that cross the threshold voltage, V_{th}, per unit time is the instant mean firing rate of this neuronal population, $r(t)$. The neurons whose states have been across V_{th} re-enter the state space via the resetting voltage V_r. The ensemble average, V, represents the macroscopic state of this neuronal population.

2.1. Leaky integrate-and-fire (LIF) model for cortical pyramidal neurons

The LIF model is formulated by[17]

$$C\dot{V} = -g_l(V - E_l) + S(t)$$
$$\text{if } V \geq V_{th}, \text{ then } V \to V_r \tag{1}$$

where C is the capacitance of the neuronal membrane; $V(t)$ is the membrane potential of neurons; E_l and g_l are respectively the constant of reversal potential and conductance of the leakage channel. In this model, the time variation of V is affected by the leakage current (the first term on the right side) and the current $S(t)$, that is, so-called post-synaptic current, induced by the presynaptic spike train. V is immediately reset to the resetting voltage, V_r, if it reaches the threshold voltage, V_{th}, and a spike is marked at that time moment.

Fig. 1 explains idea of the population density method using the LIF model as an example. The basic concept of the population density method comes from statistical mechanics [4], especially that for describing macroscopic states of an ensemble of N-particles [18]. Imagine that the alteration in the states of neurons within an individual neuronal population corresponds to the movement of particles in a space. Thus PDF, $\rho(\mathbf{x}, t)$, represents the probability density of neurons' clustering at a given state point, $\mathbf{x}$, within an enclosed state space and at a specific time moment, t. Then, the macroscopic states of a neuronal population can be obtained using ensemble averages if its PDF is known and its dynamics is the time variation of those macroscopic states. The PDF must be known as time goes so it is necessary to solve the governing equation that describes time variation of the PDF and it is derived from conservation of total number of neurons within an individual neuronal population.

For a population of cortical pyramidal neurons, the governing equation of the PDF, $\rho(V,t)$, based on the LIF model and ignorance of the dynamics of $S(t)$ is given by [4]

$$\frac{\partial}{\partial t}\rho(V,t) = -\frac{\partial}{\partial V}\left[\frac{-1}{\tau}(V - E_l)\rho\right] - \sigma(t)\rho(V,t)$$
$$+\sigma(t)\rho(V-\varepsilon,t) + r(t)\delta(V - V_r), \tag{2}$$

where time constant $\tau = C/g_l$, $\sigma(t)$ (unit: pulse per second, pps) is the time-dependent mean rate of presynaptic spike train to each neuron and ε is the voltage change due to a presynaptic spike. The mean firing rate, $r(t)$, of the neuronal population is defined as the number of neurons whose membrane potentials cross the