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Estimating the ice thickness of shallow glaciers from surface topography and mass-balance data with a shape optimization algorithm

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ABSTRACT

A shape optimization algorithm is presented that estimates the ice thickness distribution within a three-dimensional, shallow glacier, given a transient surface geometry and a mass-balance distribution. The approach is based on the minimization of the surface topography misfit in the shallow ice approximation by means of a primal-dual procedure. The method's essential novelty is that it uses surface topography and mass-balance data only within the context of a time-dependent problem with evolving surface topography. Moreover, the algorithm is capable of computing some of the model parameters concurrently with the ice thickness distribution. The method is validated on synthetic and real-world data, where the choice of its Tikhonov regularization parameter by means of an L-curve criterion is discussed.

1. Introduction

While complete digital elevation models (DEM) of surface topography can be obtained accurately by geophysical means, bedrock topography measurements can only be performed at selected locations and for a limited sample of glaciers (Christensen et al., 2000). Moreover, ice rheology, surface mass-balance, and basal sliding depend on parameters whose physical values can hardly be measured precisely and are therefore still the subjects of current research (Gudmundsson, 1999; Huss et al., 2008; Avdonin et al., 2009; Huss et al., ; Arthern and Gudmundsson, 2010; Jouvet et al., 2011; Habermann et al., 2012; Petra et al., 2012; Budd et al., 2013). In this paper, focus is put on the computation of a glacier's subglacial topography from surface topography and mass-balance measurements. Since the surface topography is known, the problems of determining the glacier's ice thickness or basal topography are equivalent.

Direct methods based on the perfect plasticity assumption deduce the ice thickness from a basal yield stress (Haeberli and Hoelzle, 1995; Li et al., ; Paul and Linsbauer, 2012). Alternative procedures relying on mass turnover and parallel-sided slabs or shallow glaciers (Hooke, 2005; Greve and Blatter, 2009; Paterson and Cuffey, 2010) supply an ice thickness distribution from analytical inversions (Farinotti et al., 2009; Michel et al., 2013b). Moreover, the recently introduced transient inverse method that iteratively updates the ice thickness with the surface topography discrepancy successfully inverts surface topography data (van Pelt et al., 2013; Michel et al., 2013b). Literature is also abundant on methods aiming at determining a glacier's basal properties, namely basal topography and sliding, from surface velocity measurements (Gudmundsson et al., 2001; Thorsteinsson et al., 2003; Raymond and Gudmundsson, 2009; Raymond-Pralong and Gudmundsson, 2011; Farinotti et al., 2012; McNabb et al., 2012; Gessese, 2013). The algorithm devised in this contribution improves the bedrock topography update of the transient inverse method by taking into account the non-locality of bedrock-to-surface topography perturbation transfers due to ice flow dynamics. Moreover, it combines subglacial topography reconstruction with computation of the model parameters.

The aforementioned direct inversion algorithms have the advantage of being easily implementable. Nevertheless, most of them are slow, especially in three space dimensions. Therefore, a

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more efficient, gradient-based method is here tailored for three-dimensional, sliding, transient, shallow glaciers. Its flexibility allows to simultaneously compute both the ice thickness distribution and ice rheology and surface mass-balance parameters. Although its implementation requires much more coding and computations by hand than the existing direct methods, this versatile procedure has the convenience of being more accurate. This preliminary study highlights the possibilities of the method and improves targeting the search for new algorithms for the Stokes model (Gudmundsson, 1999; Zwinger et al., 2007; Jouvet, 2010; Jouvet et al., 2009), in which case further complications arise, namely due to meshing issues. Many algorithms already exist for the shape optimization of time-independent PDE-constrained problems (Abe and Koro, 2006; Allaire et al., 2004). Literature is however less profuse on such methods aiming at PDE-constrained problems where the PDE is a time-dependent, free surface equation (see *e.g.* Kasumba and Kunisch, 2012).

In the simplified context of time-dependent, free surface, shallow ice flows, a primal-dual method (*e.g.* Becker et al., 2000; Nocedal and Wright, 2006; Goldberg and Sergienko, 2011) is advocated to solve the control problem in a “first discretize, then optimize” approach (Hinze et al., 2009). The forward model equations are first discretized with finite differences and then differentiated with respect to the bedrock topography elevation at each grid point. This has the advantage of providing the exact gradient of the investigated discrete objective function and a fully converging optimization process. The amount of coding could be reduced with the use of automatic differentiation (Heimbach and Bugnion, 2009; Roth and Ulbrich, 2013), but this is not the focus of the following considerations.

Hereafter, the three-dimensional, shallow ice model is first recalled. Then, the related shape optimization algorithm is elaborated. From its most basic expression that only computes the bedrock topography, a more advanced formulation is proposed that concurrently infers some model parameters and accounts for possibly available surface velocities, surface topographies, and measured bedrock topography profiles. Next, numerical results are presented, including the application of the method to real-world data. Finally, general conclusions are drawn.

2. Forward model

A three-dimensional, time-dependent glacier ice volume $\Omega(t)$ enclosed in a cavity $\Lambda = \Omega_\perp \times [\underline{Z}, \overline{Z}]$, with $\Omega_\perp = [0, L_x] \times [0, L_y]$ (see Fig. 1), is considered. Henceforth, the time-independent domain $\Omega_\perp$ is termed “glacier map domain.” The ice volume is bounded by the glacier's bedrock and surface topographies, b and s respectively, which are both functions of the horizontal coordinates x and y. Glacier isostasy and erosion are neglected, hence only the surface topography actually changes with time. Accordingly, the ice thickness is defined by $\mathcal{H} = s - b$. At initial time t_i, the surface topography is given and denoted by s_i. The forward model's purpose is to compute the surface topography at final time t_f. The final surface topography is denoted by $s_f = b + \mathcal{H}|_{t=t_f}$.

Ice is a non-Newtonian, incompressible fluid of extremely large viscosity compared to typically encountered fluids. For example, it is about 10^{16} times more viscous than temperate water. Such a high viscosity makes ice move very slowly, of the order of magnitude of 100 m per year (m a^{-1} in this paper) in Swiss Alps glaciers. In this paper, ice dynamics is assumed to be governed by the three-dimensional shallow ice approximation of flow (Hutter, 1983; Morland, 1984), which is the zero-th order approximation of the Stokes ice flow (Meur et al., 2004; Gudmundsson, 1999; Zwinger et al., 2007). Ice rheology is characterized by Glen's flow law (Glen, 1958) and the glacier's surrounding climate is encoded in the so-called surface mass-balance function $\mathcal{B}$, a representation of which is given on Fig. 1. Basically, two regions of the glacier's surface are distinguished where snow accumulates or ice melts, delimited by the so-called equilibrium line altitude (ELA).

2.1. Continuous equations

In the shallow ice approximation, the flow regime is essentially a simple, bed-parallel shear. Consequently, the ice velocity components can be expressed as the following analytical functions of the ice thickness $\mathcal{H}$ and the glacier's surface topography $s = b + \mathcal{H}$ (Greve and Blatter, 2009):

$$u_x(x,y,z) = -\left(\Gamma_s(x,y)\mathcal{H}^n(x,y) + \frac{n+2}{n+1}\Gamma(\mathcal{H}^{n+1}(x,y) - (s(x,y)-z)^{n+1})\right) \times \|\nabla s(x,y)\|^{n-1}\frac{\partial s}{\partial x}(x,y) \tag{1}$$

$$u_y(x,y,z) = -\left(\Gamma_s(x,y)\mathcal{H}^n(x,y) + \frac{n+2}{n+1}\Gamma(\mathcal{H}^{n+1}(x,y) - (s(x,y)-z)^{n+1})\right) \times \|\nabla s(x,y)\|^{n-1}\frac{\partial s}{\partial y}(x,y) \tag{2}$$

$$u_z(x,y,z) = -\int_b^z \left(\frac{\partial u_x}{\partial x}(x,y,\overline{z}) + \frac{\partial u_y}{\partial y}(x,y,\overline{z})\right) d\overline{z}, \tag{3}$$

where

$$\Gamma = 2\frac{A(\rho g)^n}{n+2} \quad \text{and} \quad \Gamma_s = (\mathcal{C}\rho g)^n [z_{sl} - b]^+ \tag{4}$$

are diffusion coefficients, A is the rate factor, ρ the ice density, g the acceleration due to gravitation, $n \geq 1$ Glen's flow law exponent (Glen, 1958), $\mathcal{C}$ a positive real constant, z_{sl} the altitude below which sliding occurs, and $[z_{sl} - b]^+$ the positive part of $z_{sl} - b$, *i.e.*

$$[z_{sl} - b]^+ = (z_{sl} - b)\vartheta(z_{sl} - b), \tag{5}$$

where ϑ is the Heaviside function

$$\vartheta(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

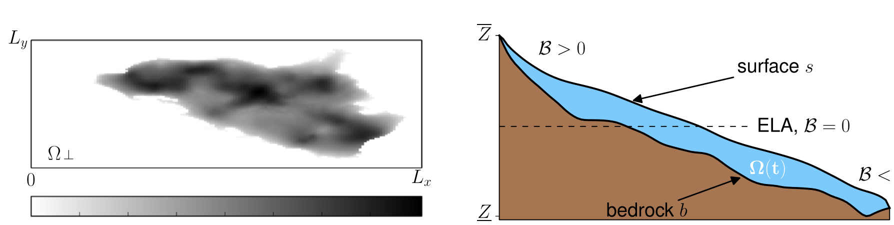

Fig. 1. Left: Ice extent in the (x, y)-plane of Silvretta glacier, Swiss Alps. The gray levels represent ice thickness distribution (with colorbar given in meters). The map domain $\Omega_\perp$ consists in the whole rectangle $[0, L_x] \times [0, L_y]$. Right: glacier profile along a flow line. Ice is represented in blue, rock (or lithosphere) in brown, and air in white. The bedrock topography b constitutes the ice – lithosphere interface, while the surface topography s the ice – air interface. The ice domain is depicted in blue. The surface mass-balance $\mathcal{B}$ is also illustrated. Above the equilibrium line altitude (ELA), snow accumulates ($\mathcal{B} > 0$), while ice melts below it ($\mathcal{B} < 0$). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)