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On a generalized Love's problem

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ABSTRACT

We present explicit expressions for computing the displacements induced in a homogeneous, linearly elastic half-space by uniform vertical pressure applied over an arbitrary polygonal region of the horizontal surface. By suitably applying Gauss theorem and recent results of potential theory we derive formulas which allow one to evaluate the displacements at an arbitrary point of the half-space solely as a function of the position vectors of the boundary of the loaded region assumed to be polygonal. Representative numerical examples referred to geodetically observed elastic displacements of the Earth surface due to water loads show the effectiveness and the flexibility of the proposed approach. Actually, it allows for a more realistic evaluation of displacements distribution and to achieve a considerable simplification in data handling since it is now possible to avoid tiling of complex regions by the simple load shapes, such as circles or rectangles, for which analytical solutions are currently available in the literature.

1. Introduction

The surface of the Earth exhibits seasonal displacements in response to annual and interannual changes in atmospheric and seafloor pressure and, even more importantly, to water loads such as those associated with shifting masses of snow, ice, surface and subsurface water (Mangiarotti et al., 2001; Van Dam et al., 2001; Dong et al., 2002; Elósegui et al., 2003).

The elastic response of the solid Earth surface due to environmental loadings can manifest at global (Blewitt et al., 2001), regional (Heki, 2001) and local scales (Bevis et al., 2004). Nowadays it can be measured with sufficient spatial and temporal resolution thanks to the rapid growth of geodetic reference networks of GNSS.

For instance Davis et al. (2004) used space-based measurements of gravity change, geodetic measurements from ten GPS stations and a global elastic model to estimate the annual deformation of South America.

Bevis et al. (2005) obtained a fairly good fit to the vertical displacement history at the geodetic GPS station located in the city of Manaus (Brazil) and showed that the response was dominated by surface loads with a horizontal scale of approximately 100 km. The surface displacements were sensitive to the elastic structure of the crust and the lithospheric mantle but insensitive to that of the mesosphere and of the lower mantle.

More recently Steckler et al. (2010) analyzed the large surface load due to the unpounded water in Bangladesh caused by the discharge of the major rivers during the summer monsoon. In particular they successfully compared the elastic deformation of the lithosphere at the annual timescales with that recorded by continuous GPS stations as a seasonal vertical deflection that can reach 5–6 cm.

The studies quoted above have proven that, when loads are applied for time periods of approximatively 1 year or less, inelastic effect can be ignored in modeling the deformation of the solid Earth caused by surface loads of global or regional extent.

Furthermore, when considering loads of limited spatial extent, i.e. those having a small spatial scale compared to the radius of the Earth, it is quite reasonable to ignore the curvature and the topography of the Earth and to consider it as an elastic half-space (Boussinesq, 1885; Love, 1929).

Thus, the elastic half-space model does represent a valuable tool since it allows one to obtain an analytical solution for non-trivial loading geometries (Becker and Bevis, 2004) and, at the same time, it turns out to be extremely efficient from the computational standpoint.

Moreover, the elastic half-space solution is useful as a limiting or special case of more sophisticated and more general Earth models (Mooney et al., 1998) such as those incorporating surface curvature and radial variation in its elastic structure (Dziewonski and Anderson, 1981; Farrell, 1972; Guo et al., 2004).

In any case one should keep in mind that the accuracy of the results obtained by more refined Earth models, such as the spherically symmetric, non-rotating, elastic isotropic (SNREI) one, can be

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hampered by the need of homogenizing somehow (Guo et al., 2004) the elastic properties of the ocean layer and of the uppermost crustal layer so as to comply with the assumptions of the PREM model by Dziewonski and Anderson (1981).

In addition, the Earth model based on a uniform elastic half-space is very useful when displacement observations are limited to a single point and only the vertical component of displacements is very well resolved, e.g. Bevis et al. (2005). This precludes the adoption of a multilayered elastic model since properties of the layers cannot be inferred by the observed response to annual loading cycles.

On the other hand the radially symmetric PREM model (Dziewonski and Anderson, 1981) is not accurate in the Earth's outermost 50 km due to the extreme heterogeneity of the uppermost mantle and especially the crust. Thus, more refined crustal models, based on seismic refraction surveys (Mooney et al., 1998), cannot be adopted since the relevant surveys do not exist in specific regions.

In this case the only realistic model is the half-space though its useful properties are countervailed by the necessity of accurately describing the loading region.

Surface loads on the half-space have been modeled by multiple discs (Elósegui et al., 2003) in order to take advantage of the solution contributed by Lamb (1902) and Terazawa (1916); actually, they extended to the case of circular loaded areas with uniform pressure, the celebrated solution due to Boussinesq (1885) for a point load, see Johnson (1985) for a survey account.

In particular Boussinesq expressed displacements and stresses at any point of the half-space as derivatives of suitable potentials. Subsequently the stresses within the half-space due to a vertical uniform pressure acting over a rectangular area were derived by Love (1929) while the corresponding solution in terms of displacements has appeared only recently (Becker and Bevis, 2004).

The aim of this paper is to further generalize the solution contributed by Becker and Bevis by providing formulas for computing the displacements within a homogeneous linearly elastic half-space due to a uniform vertical pressure applied on its top over a region of arbitrary polygonal shape.

This allows for a more flexible and accurate description of the loaded region with respect to the existing formulations in which nonuniform surface loads are approximated by means of discs (Elósegui et al., 2003) or rectangles (Becker and Bevis, 2004; Mojzeš et al., 2012).

The formulas derived in the paper are proven to be analytically well defined for every point of the half-space and every position of the loaded region. These properties have been confirmed numerically by running a home-made MATLAB® code, appended to the paper as supplementary material, for several examples selected from the literature.

The first example, derived from Elósegui et al. (2003), has been selected to show that a rough approximation of the loading conditions, even if associated with a more refined Earth model with respect to the half-space, can produce unrealistic displacement distributions.

The second example refers to a frequent case in the applications, see e.g. Steckler et al. (2010), in which the complexity of the loaded region obliges the analyst to subdivide it into a very fine mesh of subregions, typically of circular or rectangular shape, for which analytical solution is currently available in the literature.

Conversely, it is shown that the solution can be achieved much more easily by superimposing several polygonal loaded regions for which the same load value, associated with the water level, is assumed.

The paper is organized as follows. Section 2 briefly recalls the basic formulas which express the displacements within an elastic, isotropic half-space as derivatives of suitable potentials extended to the loaded region. Assuming that such a region has a polygonal shape, Sections 3 and 4 report the formulas for computing, respectively, the horizontal and vertical displacements.

All the analytical details have been included in the Appendix, where the interested reader can find the mathematical justification of the derived formulas.

Finally, Section 5 contains the results of some of the numerical examples, run by the above mentioned MATLAB® code, which have been run in order to validate the proposed approach.

2. Problem definition

Let us consider a homogeneous isotropic half-space and an orthonormal reference frame (O,x,y,z) with the z-axis directed downwards; the boundary of the half-space is defined by the plane $z = 0$.

Given a constant vertical pressure p_z applied on a portion of the plane $z = 0$, our aim is to evaluate the displacements induced at an arbitrary point $\mathbf{p} = (x,y,z)^t$ of the half-space.

This problem has been first solved by Boussinesq (1885) for a point load acting on the boundary of the half-space by expressing the solution as a function of suitable scalar potentials. Subsequently, according to a footnote in the paper by Love (1929), it has been reformulated by Hertz (1881) in the form addressed hereafter.

Recently Becker and Bevis (2004) have evaluated the stress state induced by a vertical pressure over a rectangular region by combining the formulas derived by Hertz, to express the displacements in a half-space induced by a vertical point load on its boundary, with those proposed by Love.

Specifically, the solution presented by Becker and Bevis (2004) is expressed in a form identical to that contributed by Hertz:

$$\mathbf{d}(\mathbf{p}) = -\frac{1}{4\pi}\begin{pmatrix} \dfrac{1}{\lambda+\mu}\dfrac{\partial U(\mathbf{p})}{\partial x} + \dfrac{z}{\mu}\dfrac{\partial V(\mathbf{p})}{\partial x} \\ \dfrac{1}{\lambda+\mu}\dfrac{\partial U(\mathbf{p})}{\partial y} + \dfrac{z}{\mu}\dfrac{\partial V(\mathbf{p})}{\partial y} \\ \dfrac{z}{\mu}\dfrac{\partial V(\mathbf{p})}{\partial z} - \dfrac{\lambda+2\mu}{\lambda\mu+\mu^2}V(\mathbf{p}) \end{pmatrix} \tag{1}$$

apart from the introduction of the potentials:

$$U(\mathbf{p}) = \int_\Omega p_z(\mathbf{r}) \ln(z+|\mathbf{r}-\mathbf{p}|)\, d\Omega, \quad V(\mathbf{p}) = \int_\Omega \frac{p_z(\mathbf{r})}{|\mathbf{r}-\mathbf{p}|}\, d\Omega \tag{2}$$

They represent the integrals of Boussinesq's potentials extended to the loaded area Ω and are expressed in terms of the modulus $|\cdot|$ of the vector $\mathbf{r}-\mathbf{p}$. In turn $\mathbf{r} = (x',y',0)^t$ represents a generic point of the plane $z = 0$ on which the load is applied.

In formula (1) $\mathbf{d}$ represents the displacement vector while λ and μ are the two Lamé constants; they are related to Young's modulus E and Poisson's ratio ν by the well-known expressions (Turcotte and Schubert, 1982; Kennet and Bunge, 2008):

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \tag{3}$$

To emphasize the fact that the integrals in (2) need to be carried out for a two-dimensional domain, we denote the 2D vector by a Greek letter:

$$\boldsymbol{\rho} = (x-x', y-y')^t \tag{4}$$

so that, for a uniform vertical pressure p_z, the potentials U and V can be equivalently written as follows:

$$U(\mathbf{p}) = p_z\int_\Omega \ln\left(z+\sqrt{\boldsymbol{\rho}\cdot\boldsymbol{\rho}+z^2}\right) d\Omega, \quad V(\mathbf{p}) = p_z\int_\Omega \frac{d\Omega}{\sqrt{\boldsymbol{\rho}\cdot\boldsymbol{\rho}+z^2}} \tag{5}$$