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Alternative transformation from Cartesian to geodetic coordinates by least squares for GPS georeferencing applications

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ABSTRACT

The inverse transformation of coordinates, from Cartesian to curvilinear geodetic, or symbolically $(x,y,z) \rightarrow (\lambda,\varphi,h)$ has been extensively researched in the geodetic literature. However, published formulations require that the application must be deterministically implemented point-by-point individually. Recently, and thanks to GPS technology, scientists have made available thousands of determinations of the coordinates (x,y,z) at a single point perhaps characterized by different observational circumstances such as date, length of occupation time, distance and geometric distribution of reference stations, etc. In this paper a least squares (LS) solution is introduced to determine a unique set of geodetic coordinates, with accompanying accuracy predictions all based on the given sets of individual (x,y,z) GPS-obtained values and their variance–covariance matrices. The (x,y,z) coordinates are used as pseudo-observations with their attached stochastic information in the LS process to simultaneously compute a unique set of (λ,φ,h) curvilinear geodetic coordinates from different observing scenarios.

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1. Introduction

Many scientists have investigated the so-called, non-trivial, inverse transformation of coordinates from Cartesian (x,y,z) to curvilinear (orthogonal) geodetic coordinates (λ,φ,h). Both sets of coordinates are defined with respect to any arbitrary geodetic Cartesian reference frame and, in the case of geodetic coordinates, a complementary rotational ellipsoid with center at the origin of the Cartesian frame, its semi-minor axis coincident with the z-axis, and its semi-major axis on the equatorial plane defined by the x and y axes. The selected rotational ellipsoid is typically the GRS80 ellipsoid as adopted by the International Association of Geodesy (Moritz, 1992). For an exhaustive study of the many inverse transformations available to the user, the readers may consult Featherstone and Claessens (2008) and Awange et al. (2010, p. 157) where they will find a partial list of approaches to solve this specific transformation problem. A recent article by Shu and Li (2010) cites newly developed algorithms to compute geodetic coordinates not mentioned in any of the above mentioned references. For completeness, it should also be mentioned that the International Earth Rotation and Reference System Service (IERS) recommends the use of Fukushima's (1999) iterative method. However, all these transformation equations and algorithms were analytically developed to compute coordinates in a one-by-one point basis, that is, given the Cartesian coordinates of a point determine the equivalent curvilinear geodetic coordinates at the same point; therefore they are deterministic methods but not stochastic methods.

The alternative method presented herein takes advantage of the large number of (x,y,z) determinations that we sometimes have on hand these days at a single particular point when processing GPS data. The main idea that we are proposing is to obtain the most accurate curvilinear geodetic coordinates obtainable from the complete set of available GPS "detrended" time series (x,y,z) coordinates (e.g. Teza et al., 2010). Not only are the results going to be statistically meaningful, in a least squares (LS) sense, because as a byproduct the full variance–covariance (v–c) matrix for this uniquely derived triplet of coordinates can be determined. This statistical element is missing from the standard transformation formulas mentioned above. Other options, as for example, taking the weighted mean of all individually computed (x,y,z) values and finally transform them to (λ,φ,h) using any of the current methods will not be as complete, statistically speaking, as the procedure that will be outlined in this paper. If nothing else, because the v–c matrix of each individually processed GPS point (x,y,z) is known and taken into consideration in the LS solution. This information is not properly exploited when taking any other type of statistical sample averages.

As an immediate practical application of this procedure, one may think of the calculation of a single unique set of geodetic coordinates for a point, referred to a predefined datum ellipsoid, determined from a set of original GPS-processed (x,y,z) solutions. The intention here is to get the "best" (λ,φ,h) coordinates for each

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point that, subsequently, could be stored in the type of geodetic databases that are constantly revised and updated because of the introduction of new GPS positioning data.

Present-day GPS applications allow for the archiving of (x,y,z) coordinates of the same point perhaps determined at different times by many practitioners using a diverse spectrum of observing session durations (e.g. 1 h vs. 2 h, etc.). However, it is known that the accuracy of GPS processed solutions is highly dependent on the total time span of the observing session (Eckl et al., 2001; Soler et al., 2006); therefore, the input v–c matrix of the Cartesian coordinates implicitly contains the quality of the (x,y,z) pseudo-observations.

The recommended methodology can employ one and every one or all of these thousands of independent sets of (x,y,z) coordinates at a single point (perhaps obtained from different GPS campaigns) with their available stochastic model to compute, through a LS procedure, a unique value for the curvilinear geodetic coordinates at the same point making full use of the available statistics, which, as is well known, are highly dependent on many factors, such as total observation time span, atmospheric conditions, etc.

At the end, if required, a straight one-by-one direct transformation from geodetic to Cartesian (see Eq. (1) below) could be implemented by adding the corresponding statistics after transforming the final v–c matrix of the curvilinear coordinates just determined by LS to the Cartesian v–c matrix.

2. Mathematical formulation

The basic mathematical relationship between Cartesian and orthogonal curvilinear geodetic coordinates is attributed to Helmert (1880, p. 136) and can be written in matrix form as (see e.g. Soler, 1976):

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} (N+h)\cos\varphi\cos\lambda \\ (N+h)\cos\varphi\sin\lambda \\ (N(1-e^2)+h)\sin\varphi \end{Bmatrix} \tag{1}$$

where

$$N = \frac{a}{W} \text{ is the principal radius of curvature along the prime vertical} \tag{2}$$

$$M = \frac{a(1-e^2)}{W^3} \text{ is the principal radius of curvature along the meridian} \tag{3}$$

$$W^2 = 1-e^2\sin^2\varphi \tag{4}$$

$$e^2 = 2f-f^2 \tag{5}$$

In all the above equations, a and f are the semi-major axis and flattening of the selected ellipsoid, respectively.

2.1. Least squares methodology

The theory described herein follows the mathematical reasoning and matrix notation described in Leick (2004, p. 110).

The general functional expression (mathematical model) used in the adjustment is given by Eq. (1) and can be written symbolically as

$$\ell_a = \mathbf{f}(\mathbf{x}_a) \tag{6}$$

where ℓ_a denotes the vector of n adjusted observations and $\mathbf{x}_a$ denotes u adjusted parameters (unknowns).

From Eq. (1) we can write explicitly the required set of variables as follows:

The column matrix containing the unknowns ($u=3$) is

$$\underset{3\times 1}{\mathbf{x}_a} = \begin{Bmatrix} \lambda \\ \varphi \\ h \end{Bmatrix} \tag{7}$$

while the column matrix of observables takes the form:

$$\underset{3n\times 1}{\ell_b} = \begin{Bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_1 \\ \vdots \\ \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_n \end{Bmatrix} \tag{8}$$

Notice that the total number of repeated observations of the same point is n, and that the total number of equations in (6) is $r=3n$.

According to Leick (2004, p. 111), the least squares solution of the adjustment model can be written in matrix notation as

$$\underset{3\times 1}{\hat{\mathbf{x}}} = -\mathbf{N}^{-1}\mathbf{u} \tag{9}$$

where the normal equation matrices are given explicitly by

$$\underset{3\times 3}{\mathbf{N}} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A} \tag{10}$$

and

$$\underset{3\times 1}{\mathbf{u}} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\ell \tag{11}$$

The design matrix $\mathbf{A}$ in Eqs. (10) and (11) is computed according to the following matrix expression:

$$\underset{3n\times 3}{\mathbf{A}} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_0,\ell_b} = \begin{bmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial\lambda} & \frac{\partial f_1}{\partial\varphi} & \frac{\partial f_1}{\partial h} \\ \frac{\partial f_2}{\partial\lambda} & \frac{\partial f_2}{\partial\varphi} & \frac{\partial f_2}{\partial h} \\ \frac{\partial f_3}{\partial\lambda} & \frac{\partial f_3}{\partial\varphi} & \frac{\partial f_3}{\partial h} \end{bmatrix}_1 \\ \begin{bmatrix} \frac{\partial f_4}{\partial\lambda} & \frac{\partial f_4}{\partial\varphi} & \frac{\partial f_4}{\partial h} \\ \frac{\partial f_5}{\partial\lambda} & \frac{\partial f_5}{\partial\varphi} & \frac{\partial f_5}{\partial h} \\ \frac{\partial f_6}{\partial\lambda} & \frac{\partial f_6}{\partial\varphi} & \frac{\partial f_6}{\partial h} \end{bmatrix}_2 \\ \vdots \\ \begin{bmatrix} \frac{\partial f_{r-2}}{\partial\lambda} & \frac{\partial f_{r-2}}{\partial\varphi} & \frac{\partial f_{r-2}}{\partial h} \\ \frac{\partial f_{r-1}}{\partial\lambda} & \frac{\partial f_{r-1}}{\partial\varphi} & \frac{\partial f_{r-1}}{\partial h} \\ \frac{\partial f_r}{\partial\lambda} & \frac{\partial f_r}{\partial\varphi} & \frac{\partial f_r}{\partial h} \end{bmatrix}_n \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -(N+h)\cos\varphi\sin\lambda & -(M+h)\sin\varphi\cos\lambda & \cos\varphi\cos\lambda \\ (N+h)\cos\varphi\cos\lambda & -(M+h)\sin\varphi\sin\lambda & \cos\varphi\sin\lambda \\ 0 & (M+h)\cos\varphi & \sin\varphi \end{bmatrix}_1 \\ \begin{bmatrix} -(N+h)\cos\varphi\sin\lambda & -(M+h)\sin\varphi\cos\lambda & \cos\varphi\cos\lambda \\ (N+h)\cos\varphi\cos\lambda & -(M+h)\sin\varphi\sin\lambda & \cos\varphi\sin\lambda \\ 0 & (M+h)\cos\varphi & \sin\varphi \end{bmatrix}_2 \\ \vdots \\ \begin{bmatrix} -(N+h)\cos\varphi\sin\lambda & -(M+h)\sin\varphi\cos\lambda & \cos\varphi\cos\lambda \\ (N+h)\cos\varphi\cos\lambda & -(M+h)\sin\varphi\sin\lambda & \cos\varphi\sin\lambda \\ 0 & (M+h)\cos\varphi & \sin\varphi \end{bmatrix}_n \end{bmatrix} \tag{12}$$