Computers and Structures 205 (2018) 1–14

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

Variational formulation of micropolar elasticity using 3D hexahedral finite-element interpolation with incompatible modes

Sara Grbčić [a,b], Adnan Ibrahimbegović [b,*], Gordan Jelenić [a]

[a] University of Rijeka, Faculty of Civil Engineering, Croatia
[b] Université de Technologie de Compiègne – Sorbonne Universités, Chair for Computational Mechanics & Insititut Universitaire de France, France

ARTICLE INFO

Article history:
Received 14 February 2018
Accepted 26 April 2018

Keywords:
Micropolar theory
Finite element method
Incompatible modes
Model validation

ABSTRACT

A three-dimensional micropolar elasticity is cast in terms of the rigorous variational formulation. The discrete approximation is based on hexahedral finite element using the conventional Lagrange interpolation and enhanced with incompatible modes. The proposed element convergence is checked by performing patch tests which are derived specifically for micropolar finite elements. The element enhanced performance is also demonstrated by modelling two boundary value problems with analytical solutions, both exhibiting the size-effect. The analyzed problems involve a cylindrical plate bending and pure torsion of circular cylinders, which were previously used in the experimental determination of the micropolar material parameters. The numerical results are compared against the analytical solution, and additionally against existing experiments on a polymeric foam for the pure torsion problem. The enhancement due to incompatible modes provides the needed improvement of the element performance in the bending test without negative effects in the pure-torsion test where incompatible modes are not needed. It is concluded that the proposed element is highly suitable for the numerical validation of the experimental procedure.

1. Introduction

Most of the materials are heterogeneous in general, with a specific microstructure that can be represented at a scale particular for the material itself. When this scale is very small, these materials are considered as homogeneous. For such materials (e.g. metals), any microstructure detail is averaged leading to a homogeneous continuum theory. Commonly used is Cauchy's or classical theory that is able to faithfully describe the material behavior. However, when the microstructure scale becomes significantly large compared to the overall scale, assuming the homogenized material, representation based on the classical theory fails. Many newly developed engineering materials increasingly used in engineering, such as fiber-reinforced composites, honeycomb or cellular structured materials or modern polymers belong to the last category. Due to their heterogeneity, such materials exhibit a so-called *size-effect* phenomenon, which manifests in increased stiffness of smaller specimens made of the same material, which is not recognized in the classical continuum theory. Moreover, in regions of high stress gradients, such as the neighborhoods of holes, notches and cracks, the stress concentration factor as predicted by the classical theory is higher than that observed experimentally. Even more discrepancies between the classical continuum theory and the experimental testing may be observed in dynamics, thermal analysis and fluid mechanics [1]. Due to such anomalies, an alternative continuum model to accurately describe the behavior of such materials is highly needed.

One such model, further discussed in this paper, is the so-called oriented, or Cosserat or micropolar continuum. Namely, different approaches are developed to study the multi-scale nature of the material deformation, by taking into account additional effects consistent with the observed behavior of such heterogeneous materials. One such development accounting for microstructure effects within the limits of continuum mechanics is introducing *higher order derivatives* or the field gradients, such as the so-called couple-stress or higher-order strain-gradient theories. An alternative approach is introducing *additional degrees of freedom*, such as micro-stretch or micro-morphic continuum theory [2], to name only a few. Among such theories introducing additional degrees of freedom, we further elaborate upon so-called *micropolar* continuum theory, usually attributed to the Cosserat brothers [3]. They enriched the Cauchy's theory by adding to the displacement field an independent *microrotation* field, representing the *local rotation* of a material point. The detailed exposition of the historical

* Corresponding author.
E-mail addresses: sara.grbcic@uniri.hr (S. Grbčić), adnan.ibrahimbegovic@utc.fr (A. Ibrahimbegović), gordan.jelenic@uniri.hr (G. Jelenić).

https://doi.org/10.1016/j.compstruc.2018.04.005
0045-7949/© 2018 Elsevier Ltd. All rights reserved.

development of such theory can be found in [2], who named it the micropolar theory of elasticity. The main goal of this work is to contribute to the further development of such a more general theory, by performing a detailed analysis of some important micropolar boundary value problems.

The ability to include local rotation extends the modeling capabilities, and allows us to take into account the intrinsic material length-scale. However, the additional capabilities come at a cost. In order to describe such a material, even when assumed to be linear elastic, homogeneous and isotropic, it requires *six independent material constants*, in contrast to only two such constants for the classical continuum. Moreover, the experimental determination of these materials parameters is much more complex, since the experimental verification and their corresponding conceptualisation and interpretation is far from straightforward. The work in [4] is the first attempt to determine all six micropolar material constants by developing experimental and analytical solutions to the boundary value problem, but without particular success in the experimental part since opposite trends between experiments and analytical predictions have been observed. However, by subsequent refinement of Gauthier's and Jahsman's proposed procedure [4], Lakes and his co-workers give the most significant contribution to devising experimental procedures to determine the micropolar material parameters in their analysis of bones [5–7], polymeric foams [8–11] and metal foams [12], based upon measuring the size-effect. As an alternative to the experiments performed by Lakes and his co-workers, the micropolar parameter determination can be based on various homogenization procedures which replaces a larger-scale composite structure, or assembly of particles, by an effective micropolar continuum model. By assuming that a homogeneous Cosserat material is the best approximation of a heterogeneous Cauchy material, the six material parameters of the micropolar continuum may be determined more easily [13–16]. Several recent works of Wheel et al. [17–19] determined the material parameters of highly heterogeneous materials on a larger-scale by comparing the results of experiments and the finite element simulation.

However, the experimental verification of a micropolar material model still remains a great challenge, since a unified procedure to determine the material parameters of micropolar continuum is still lacking. We argue here that the key to understanding and developing more precise experimental procedures lies in the comprehensive numerical analysis of the solution of the corresponding boundary value problem. Such a comprehensive numerical analysis should broaden the range of problems which may be solved and open up new possibilities for the numerical simulation of experimental set-ups. Therefore, the development of the finite elements of high quality is important for the future progress and understanding of the micropolar continuum theory.

An early attempt to model the micropolar constitutive behaviour using the finite-element method is presented in [20] with more authors working on numerical solutions of the micropolar continuum using different finite elements in the linear analysis (e.g. [21–24]). Furthermore, in addition to the standard finite-element procedures, non-standard finite-element methods, such as the control-volume-based finite-element method [25,17] have been used to model micropolar finite elements.

The objective of this paper is to present one high quality element for 3D simulations. More precisely, we propose a high-performance three-dimensional micropolar hexahedral finite element, using conventional Lagrange interpolation enhanced with the so-called *incompatible modes* [26,27]. The proposed element performance is tested against the analytical boundary value problems derived by Gauthier and Jahsman [4] and experiments performed by Lakes and co-workers [5–12]. In the framework of the classical elasticity the incompatible displacement modes are first added to the isoparametric elements (e.g. see [26–29]). The main benefit of incompatible modes in the classical continuum framework is to avoid shear locking, as shown already in early 1970s [30]. In bending of isoparametric 4-node 2D or 8-node 3D finite elements, the absence of quadratic polynomials in the displacement field approximation predicts the shear strain in pure bending incorrectly. This is called the *shear-locking effect* [31]. Even with higher-order elements producing better results in pure-bending tests, the maximum possible reduction of computational cost is always a worthwhile goal. The proposed solution is to enrich the displacement interpolation of the corresponding element with quadratic displacement interpolation modes, requiring internal element degrees of freedom and leading to incompatibility of the displacement field. When first introduced into 2D quadrilateral isoparametric finite elements [30], the method was received with skepticism in the finite element method research community, since the displacement compatibility between finite elements was at that time considered to be absolutely mandatory [32]. The use of the incompatible-mode method for low-order elements in both two- and three-dimensional problems is nowadays common, leading to the most impressive performance not only in bending, but also elsewhere, e.g. when modelling cracking [33,29] and two-phase materials [34]. A detailed exposition of 1D, 2D and 3D finite elements with incompatible modes in classical elasticity is presented in [35].

In the framework of micropolar elasticity, the idea of enhancing the displacement field of standard finite element is already recognised in [36], where authors analyzed straight and curved beam problems subject to shear loading. Only 2D problems have been analyzed in [36] and the numerical results have not always converged to the reference analytical solution. In the present work, the high performance of the presented finite element is demonstrated by successful analysis of both 2D and 3D problems. Moreover, our ability to deliver the solution that can converge to reference values was confirmed for both bending and torsion.

2. Micropolar continuum model formulation

The fundamental relations of linear micropolar elasticity applied to a homogeneous and isotropic material are outlined in this section. We consider a continuous body $\mathcal{B}$, of volume V and boundary surface S in the deformed state under the influence of external actions consisting of distributed body force $\mathbf{p}_v$ and body moment $\mathbf{m}_v$ and distributed surface force $\mathbf{p}_s$ and surface moment $\mathbf{m}_s$. By generalising the Cauchy stress principle (see [37]), at an internal material point X, with the position vector $\mathbf{x}$, with respect to a chosen spatial frame of reference at time t, we prove the existence of a second-order Cauchy stress tensor $\boldsymbol{\sigma}(\mathbf{x}, t)$ and an additional second-order couple-stress tensor $\boldsymbol{\mu}(\mathbf{x}, t)$.

2.1. Equilibrium equations

By analysing the static equilibrium of a differential volume dV in the deformed state, we can obtain the force equilibrium equation

$$\boldsymbol{\sigma}\nabla + \mathbf{p}_v = \mathbf{0}, \tag{1}$$

where ∇ is the differential operator nabla (e.g. see [29]), and the moment equilibrium equation

$$\boldsymbol{\mu}\nabla + \mathbf{a} + \mathbf{m}_v = \mathbf{0}. \tag{2}$$