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A new enriched 4-node 2D solid finite element free from the linear dependence problem

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ABSTRACT

In this paper, we propose a new 4-node 2D solid finite element enriched by interpolation cover functions. Instead of using the bilinear shape functions of the standard 4-node finite elements, piecewise linear shape functions are adopted as the partition of unity functions to resolve the linear dependence problem; thus, rank deficiency of the stiffness matrix is not observed. Higher order cover functions can be arbitrarily employed to increase solution accuracy without mesh refinements or introduction of additional nodes. The new enriched 4-node element also shows good convergence behavior, even when distorted meshes are used. Herein, we investigate the linear dependence problem of the new enriched element. Its convergence, effectiveness, and usefulness are demonstrated through the solution of four plane stress problems: an ad hoc problem, a tool jig problem, a slender beam problem, and an automotive wheel problem.

1. Introduction

The finite element method has advantages in that it can effectively consider complicated geometries using meshes, and has been widely used for solid, fluid, and multi-physics problems. However, the accuracy of solutions depends on the quality of the meshes used, and in engineering practice, it takes considerable effort to obtain a suitable mesh. Also, mesh refinements are often necessary to secure reliable solutions with required accuracy when non-smooth, near-singular, and high-gradient solutions are sought [1,2].

In order to obtain accurate solutions, special enrichment functions incorporated within finite element formulations has been developed, where the solution space is built by multiplying the partition of unity functions by local approximation functions. The mathematical background of this technique was established by Babuška and Melenk [3]. Belytschoko and Black [4], Moes et al. [5], Dolbow et al. [6] and Daux et al. [7] applied such enrichment functions to account for discontinuities and singularities in solid mechanics problems, and Ham and Bathe [8] successfully incorporated harmonic functions to analyze wave propagation problems. The enriched finite element method based on the use of the interpolation cover functions was also studied for analysis of solids and shells by Kim and Bathe [9] and Jeon et al. [10].

In the enriched finite element methods, the shape functions of the standard finite elements and polynomial functions have been widely adopted as the partition of unity functions and as local approximation [3–6] (or interpolation cover [8–10]) functions, respectively. The methodology is simple and effectively provides overall solution improvement in the sense of the *p*-version of the finite element method, without introducing additional nodes. Furthermore, adaptively applying interpolation cover functions to local areas where the solution needs to be improved can be easily implemented. However, when both the partition of unity functions and interpolation cover functions consist of polynomials, the linear dependence (LD) problem occurs for some topologies. In this case, the global stiffness matrix becomes rank deficient even though the essential boundary conditions are properly applied. An et al. [11,12] investigated the LD problem in 2D triangular and quadrilateral elements, and in 3D hexahedron and tetrahedral elements.

There have been various attempts to resolve the LD problem. Babuška and Melenk [3] designed partition of unity functions in order that the LD problem could be overcome in 1D analysis. Oden et al. [13] suggested the elimination of linear polynomial terms in the local approximation functions. Duarte et al. [14] and Stouboulis et al. [15] showed that such treatments are not enough to avoid the LD problem; then adopted special equation solvers in 2D and 3D analyses. Tian et al. [16] studied the rank deficiency (RD) of the global stiffness matrix of enriched 2D solid elements, and found that suppressing enriched degrees of freedoms (DOFs) corresponding to enriched functions at the essential boundary is effective for 2D

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analysis with 3-node triangular elements. Tian et al. [17,18] also proposed a method to construct interpolation functions using a least squares process to resolve the LD problem. It has been well known that the LD problem can be avoided by applying the flat-top partition of unity functions, although the construction of the functions is not easy [19–22].

In this paper, we aim to develop, in a simple and effective way, a 4-node 2D solid finite element enriched by interpolation covers, which is free from the LD problem. Therefore, the resulting element makes it possible to utilize the 4-node element meshes most widely used for 2D solid mechanics problems. A set of simple shape functions (piecewise linear shape functions) is newly proposed for the enriched 4-node element. By adopting the shape functions, the LD problem is automatically resolved. The new enriched 4-node element shows good convergence behavior even when distorted meshes are used. In addition, the solution accuracy could be improved without mesh refinement or the use of higher order elements.

In the following sections, we first briefly review the enriched finite element method, and the shape functions of the new enriched 4-node element are derived. We then analytically and numerically investigate the LD problem using various mesh patterns. Through some illustrative examples, we show the convergence and computational efficiency of the new enriched 4-node element proposed in this study. Also, the adaptive use of interpolation covers is demonstrated. Finally, conclusions are drawn.

2. Finite element procedure enriched by interpolation covers

Enriching the finite element procedure, in principle, is straightforward and has a well-established mathematical background [3]. The solution space of the standard finite element method can be enriched without remeshing or introducing additional nodes [4–10,23,24]. In addition, using enrichment functions suitable for a particular problem, the solution accuracy can be effectively improved.

We here briefly review the formulation of the 4-node 2D solid finite element enriched by interpolation covers [3,9–11,16] and propose a set of shape functions for the new enriched 4-node element. In this study, linear and quadratic polynomials are considered as cover functions, leading to the enriched elements with quadratic and cubic displacement interpolations, respectively. The same principle can be directly adopted for generating enriched elements with higher order interpolations.

2.1. Formulation of the enriched 4-node solid finite element

The geometry interpolation of the enriched 4-node 2D solid finite element is identical to that of the corresponding standard finite element

$$\mathbf{x}(r,s)=\sum_{i=1}^{4}h_i(r,s)\mathbf{x}_i \quad \text{with} \quad \mathbf{x}_i=[\,x_i \quad y_i\,]^T, \tag{1}$$

where $\mathbf{x}_i$ is the position vector of node i in the global Cartesian coordinate system shown in Fig. 1(a), and $h_i(r,s)$ are the bilinear shape functions of standard isoparametric procedure corresponding to node i defined in the natural coordinate system in Fig. 1(b)

$$\begin{aligned} h_1(r,s)&=(1+r)(1+s)/4, \quad h_2(r,s)=(1-r)(1+s)/4,\\ h_3(r,s)&=(1-r)(1-s)/4, \quad h_4(r,s)=(1+r)(1-s)/4. \end{aligned} \tag{2}$$

The 2D shape functions, h_i satisfy the partition of unity requirement, $\sum_{i=1}^{4}h_i=1$. Therefore, the displacement interpolation of the enriched 4-node finite element is given by multiplying the shape functions by cover functions defined in the cover area C_i as follows [3,9,10,25]:

$$\mathbf{u}(r,s)=\sum_{i=1}^{4}h_i(r,s)\tilde{\mathbf{u}}_i \quad \text{with} \quad \tilde{\mathbf{u}}_i=[\,\tilde{u}_i \quad \tilde{v}_i\,]^T, \tag{3}$$

in which $\tilde{u}_i$ and $\tilde{v}_i$ are cover functions corresponding to the displacements in the x- and y-directions, respectively, and the cover C_i is the union of elements attached to node i (see Fig. 2).

The cover functions are given by

$$\tilde{u}_i=\mathbf{p}_i(\mathbf{x})\mathbf{u}_i^u, \quad \tilde{v}_i=\mathbf{p}_i(\mathbf{x})\mathbf{u}_i^v \quad \text{in} \quad C_i \tag{4}$$

with

$$\begin{aligned} &\mathbf{p}_i(\mathbf{x})=[1 \quad \xi_i \quad \eta_i \quad \xi_i^2 \quad \cdots \quad \eta_i^d], \quad \xi_i=\frac{(x-x_i)}{\chi_i}, \quad \eta_i=\frac{(y-y_i)}{\chi_i},\\ &\mathbf{u}_i^u=\left[u_i^1 \quad u_i^{\xi} \quad u_i^{\eta} \quad u_i^{\xi^2} \quad \cdots \quad u_i^{\eta^d}\right]^T,\\ &\mathbf{u}_i^v=\left[v_i^1 \quad v_i^{\xi} \quad v_i^{\eta} \quad v_i^{\xi^2} \quad \cdots \quad v_i^{\eta^d}\right]^T, \end{aligned} \tag{5}$$

in which $\mathbf{p}(\mathbf{x})$ is a polynomial basis vector for node i, d is the degree of polynomial bases, χ_i is the largest edge length of elements attached to node i, and $\mathbf{u}_i^u$ and $\mathbf{u}_i^v$ are the degrees of freedom (DOFs) vectors corresponding to polynomial bases for the displacements u and v, respectively.

Substituting Eq. (4) into Eq. (3), the displacement interpolation of the enriched 4-node element is obtained

$$\mathbf{u}(r,s)=\bar{\mathbf{u}}(r,s)+\hat{\mathbf{u}}(r,s)=\sum_{i=1}^{4}h_i(r,s)\bar{\mathbf{u}}_i+\sum_{i=1}^{4}\hat{\mathbf{H}}_i(r,s)\hat{\mathbf{u}}_i \tag{6}$$

with

$$\bar{\mathbf{u}}_i=\begin{bmatrix}\bar{u}_i\\ \bar{v}_i\end{bmatrix}, \quad \hat{\mathbf{u}}_i=\begin{bmatrix}\hat{\mathbf{u}}_i^u\\ \hat{\mathbf{u}}_i^v\end{bmatrix}, \quad \hat{\mathbf{H}}_i(r,s)=\begin{bmatrix}\hat{\mathbf{h}}_i(r,s) & \mathbf{0}\\ \mathbf{0} & \hat{\mathbf{h}}_i(r,s)\end{bmatrix}, \tag{7}$$

in which $\bar{\mathbf{u}}_i$ is the standard nodal displacement vector at node i in the global Cartesian coordinate system, and $\hat{\mathbf{u}}_i$ and $\hat{\mathbf{H}}_i(r,s)$ are the enriched DOFs vector and the corresponding interpolation matrix, respectively.

When the linear cover functions are used (i.e., $d=1$), the components of the interpolation matrix and the enriched DOFs vector become

$$\hat{\mathbf{h}}_i(r,s)=h_i(r,s)[\,\xi_i \quad \eta_i\,], \quad \hat{\mathbf{u}}_i^u=[\,\hat{u}_i^{\xi} \quad \hat{u}_i^{\eta}\,]^T, \quad \hat{\mathbf{u}}_i^v=[\,\hat{v}_i^{\xi} \quad \hat{v}_i^{\eta}\,]^T. \tag{8}$$

For the quadratic cover functions used ($d=2$), the following components and vector are employed

$$\begin{aligned} &\hat{\mathbf{h}}_i(r,s)=h_i(r,s)[\xi_i \quad \eta_i \quad \xi_i^2 \quad \xi_i\eta_i \quad \eta_i^2],\\ &\hat{\mathbf{u}}_i^u=\left[\hat{u}_i^{\xi} \quad \hat{u}_i^{\eta} \quad \hat{u}_i^{\xi^2} \quad \hat{u}_i^{\xi\eta} \quad \hat{u}_i^{\eta^2}\right]^T, \quad \hat{\mathbf{u}}_i^v=\left[\hat{v}_i^{\xi} \quad \hat{v}_i^{\eta} \quad \hat{v}_i^{\xi^2} \quad \hat{v}_i^{\xi\eta} \quad \hat{v}_i^{\eta^2}\right]^T. \end{aligned} \tag{9}$$

Using the displacement-strain relation, the strain vector for an element m is obtained by [9]

$$\boldsymbol{\varepsilon}^{(m)}=\mathbf{B}^{(m)}\mathbf{u}^{(m)}, \tag{10}$$

in which $\mathbf{B}^{(m)}$ is the displacement-strain relation matrix and the nodal DOFs vector $\mathbf{u}^{(m)}$ includes $\bar{\mathbf{u}}_i$ and $\hat{\mathbf{u}}_i$.

For a finite element model, the static equilibrium equations are given by

$$\mathbf{KU}=\mathbf{R}=\mathbf{R}_B+\mathbf{R}_S, \tag{11}$$

with