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A block-coupled Finite Volume methodology for linear elasticity and unstructured meshes

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ABSTRACT

The current article presents a fully coupled cell-centred Finite Volume solution methodology for linear elasticity and unstructured meshes. Details are given of the novel implicit discretisation and block coupled solution procedure, including use of a Finite Area method for face tangential gradient calculations and coupled treatment of non-orthogonal corrections. A number of 2-D and 3-D linear-elastic benchmark test cases are examined using hexahedral, tetrahedral and general polyhedral meshes; solution accuracy and efficiency are compared with that of a segregated procedure and a commercial Finite Element software, where the new method is shown to be faster in all cases.

1. Introduction

In the realm of Computational Solid Mechanics (CSM), the Finite Element (FE) method is ubiquitous; however, methods such as Finite Difference, Discrete Element, Finite Volume (FV), and so-called Meshfree methods, can provide viable alternatives in many applications. In particular, the FV method has been gradually broadening its applicability beyond Computational Fluid Dynamics and gaining momentum in the field of CSM; this may be primarily attributed to its attractively simple and strongly conservative nature, while also having a close relationship with FE methods. To date, the FV method has been applied to a large range of problems in CSM, appearing in a number of distinct forms characterised by different control volume discretisations: cell-centred [1–26], vertex-centred [27–30], and the so-called parametric formulation [31,32]. To-date, the cell-centred FV approach has typically employed a segregated solution procedure, where the momentum equation is temporally decoupled into components which are solved sequentially, with outer Picard/Fixed-Point iterations providing the coupling. This method is extremely memory efficient but can suffer from poor convergence when the inter-displacement-component coupling is strong. To overcome such inadequacies, the current article presents the development of a block-coupled solution methodology, where inter-component coupling is implicitly included as coefficients in a block matrix. Independent of the current approach, Das et al. [25] have developed a coupled cell-centred FV solution procedure; the current method shares similarities with the procedure developed by Das et al. [25], but significantly differs in a number of regards, in particular with discretisation of tangential derivatives, and the treatment of boundary conditions. Details of the differences between the two methods are given later within Section 3.

The article is constructed as follows: Section 2 outlines the mathematical model, derived from the governing momentum equation and Hookean constitutive relation, where no distinction is made between the initial and deformed configurations. The newly developed coupled cell-centred FV discretisation is presented in Section 3; the control volume face normal derivative terms are discretised using standard central differencing, and the face tangential derivative terms are discretised using the Finite Area method. Details are given of the inclusion of a larger implicitly coupled computational stencil within 3-D unstructured polyhedral code, and implementation in open-source `C++` library OpenFOAM (`foam-extend-3.1`). Section 4 presents the application of the method to three benchmark test cases, where the efficiency and accuracy of the method is compared with that of segregated FV approaches, and a commercial FE software.

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2. Mathematical model

Neglecting inertia and body forces for clarity, the conservation of linear momentum for an arbitrary body of volume Ω bounded by surface Γ with outward facing unit normal $\boldsymbol{n}$ is given in strong integral form as:

$$\int_{\Omega} \nabla \cdot \boldsymbol{\sigma}\, d\Omega = \oint_{\Gamma} \boldsymbol{n} \cdot \boldsymbol{\sigma}\, d\Gamma = 0 \tag{1}$$

The Cauchy stress tensor $\boldsymbol{\sigma}$ for a linear elastic body is given by Hooke's law:

$$\boldsymbol{\sigma} = \mu \nabla \boldsymbol{u} + \mu \nabla \boldsymbol{u}^T + \lambda \mathrm{tr}(\nabla \boldsymbol{u}) \mathbf{I} \tag{2}$$

where $\boldsymbol{u}$ is the total displacement vector, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ signifies the so called Hamilton operator, synonymous with the del or nabla operator, in the 3-D Cartesian coordinate system.

The Lamé coefficients μ and λ relating to the Young's modulus of elasticity, E, and the Poisson's ratio, ν, are given respectively as:

$$\mu = \frac{E}{2(1+\nu)} \tag{3}$$

$$\lambda = \begin{cases} \frac{\nu E}{(1+\nu)(1-\nu)} & \text{for plane stress} \\ \frac{\nu E}{(1+\nu)(1-2\nu)} & \text{for plane strain and 3-D} \end{cases} \tag{4}$$

Transient effects may be included using a standard time-marching approach, where the choice of finite difference temporal discretisation (*e.g.* backward Euler, Crank-Nicolson, Newmark-beta) will dictate the temporal accuracy and order of accuracy. Inclusion of body force terms is trivial; the use of more complex constitutive relations is also possible, but may require modifications to the solution procedure for nonlinear materials.

The mathematical model is found by substituting the constitutive relation (Eq. (2)) into the governing equation (Eq. (1)):

$$\oint_{\Gamma} \boldsymbol{n} \cdot \left[\mu \nabla \boldsymbol{u} + \mu \nabla \boldsymbol{u}^T + \lambda \mathrm{tr}(\nabla \boldsymbol{u}) \mathbf{I}\right] d\Gamma = 0 \tag{5}$$

where no distinction is made between the initial and deformed configurations *i.e.* small strains and small rotations are assumed.

3. Numerical method

The mathematical model presented in the preceding section is now discretised in a fully implicit coupled manner using the cell-centred FV method, providing a discrete approximation of the previously presented exact integral. The discretisation procedure is separated into two distinct parts: discretisation of the solution domain and discretisation of the governing equations. If temporal effects were considered, time would also be discretised into a finite number of time increments, where the mathematical model is solved in a time-marching manner.

3.1. Solution domain discretisation

The solution domain space is divided into a finite number of convex polyhedral cells bounded by polygonal faces. The cells do not overlap and fill the space completely. A typical control volume is shown in Fig. 1, with the computational node P located at the cell centroid, the cell volume is Ω_P, N is the centroid of a neighbouring control volume, face f has face area vector $\boldsymbol{\Gamma}_f$, vector $\boldsymbol{d}_f$ joins P to N and $\boldsymbol{r}$ is the positional vector of P. Note that in contrast to standard FE methods, no distinction is made between different cell volume shapes, as all general polyhedra (*e.g.* tetrahedra, hexahedra, triangular prism, dodecahedra, *etc.*) are discretised in the same general fashion: this allows greater freedom during the often troublesome mesh generation phase.

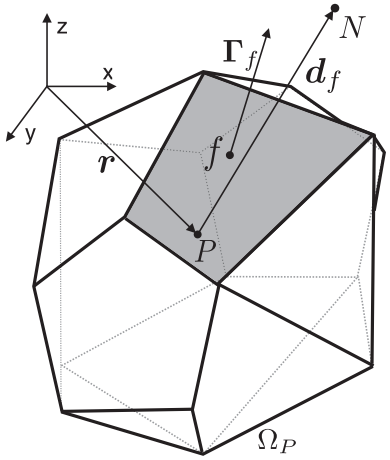

Fig. 1. General polyhedral control volume (adapted from [2,33]).

3.2. Equation discretisation

The standard cell-centred FV discretisation approach for solid mechanics, as previously presented *e.g.* [3,6,9], partitions the surface force (diffusion term) into an implicit component, included within the matrix coefficients of the resulting linear system, and an explicit component, treated in a deferred correction manner and included explicitly in the source of the linear system; this split of the diffusion term allows the three coupled scalar displacement component equations to be temporarily decoupled, allowing use of memory efficient iterative linear solvers. Outer Picard/Fixed-Point iterations provide the necessary coupling in this so-called staggered/segregated solution procedure. In contrast, the current method aims to treat the diffusion term in a fully coupled manner where only one solution of the final linear equation system is required. To facilitate this, the surface traction ($\boldsymbol{T} = \boldsymbol{n} \cdot \boldsymbol{\sigma}$) is decomposed into normal ($\boldsymbol{T}_n$) and tangential ($\boldsymbol{T}_t$) components [12]:

$$\begin{aligned} &\oint_{\Gamma} \boldsymbol{n} \cdot \left[\mu \nabla \boldsymbol{u} + \mu \nabla \boldsymbol{u}^T + \lambda \mathrm{tr}(\nabla \boldsymbol{u}) \mathbf{I}\right] d\Gamma \\ &= \oint_{\Gamma} \boldsymbol{T}\, d\Gamma = \oint_{\Gamma} [\overbrace{\boldsymbol{n}\boldsymbol{n} \cdot \boldsymbol{T}}^{\boldsymbol{T}_n} + \overbrace{(\mathbf{I} - \boldsymbol{n}\boldsymbol{n}) \cdot \boldsymbol{T}}^{\boldsymbol{T}_t}]\, d\Gamma \\ &= \oint_{\Gamma} \overbrace{(2\mu + \lambda)\boldsymbol{n} \cdot \nabla \boldsymbol{u}_n + \lambda \boldsymbol{n}\, \mathrm{tr}(\nabla_t \boldsymbol{u}_t)}^{\boldsymbol{T}_n} + \overbrace{\mu \boldsymbol{n} \cdot \nabla \boldsymbol{u}_t + \mu \nabla_t u_n}^{\boldsymbol{T}_t}\, d\Gamma \end{aligned} \tag{6}$$

where $\nabla_t = (\mathbf{I} - \boldsymbol{n}\boldsymbol{n}) \cdot \nabla$ designates a tangential derivative; quantities in bold font are vectors or tensors; subscripts n and t represent the normal and tangential components of a vector at a face, respectively. The equivalence of the second and third lines on the right-hand side of Eq. (6) are shown in expanded component form in Appendix A.

The discretisation of the normal and tangential face-derivatives are now given separately.

3.2.1. Normal derivative terms

The surface integrals may be replaced by a sum over the faces of a control volume, and assuming a linear variation of displacement $\boldsymbol{u}$ across the control volume, a second-order FV discretisation may be applied. The normal derivative terms ($\boldsymbol{n} \cdot \nabla$) on face f are discretised as follows:

$$\begin{aligned} \oint_{\Gamma} (2\mu + \lambda)\boldsymbol{n} \cdot \nabla \boldsymbol{u}_n\, d\Gamma \approx &\sum_f (2\mu_f + \lambda_f) |\boldsymbol{\Delta}_f| (\boldsymbol{n}_f \boldsymbol{n}_f) \cdot \left(\frac{\boldsymbol{u}^N - \boldsymbol{u}^P}{|\boldsymbol{d}_f|}\right) \\ &+ \sum_f (2\mu_f + \lambda_f)(\boldsymbol{n}_f \boldsymbol{n}_f) \cdot \left[\boldsymbol{k}_f (\nabla_t u_n)_f\right] \end{aligned} \tag{7}$$