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Hybrid geometric-dissipative arc-length methods for the quasi-static analysis of delamination problems

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ABSTRACT

This paper presents a class of arc-length methods for the quasi-static analysis of problems involving material and geometric nonlinearities. A constraint equation accounting for geometric and dissipative requirements is adopted: the geometric part refers to the Riks and Crisfield equations, while the dissipative one refers to the dissipated energy. The approach allows for a continuous variation of the nature of the constraint, and a switch criterion is not needed to trace the elastic and the dissipative parts of the equilibrium paths. To illustrate the robustness and the efficiency of the methods, three examples involving geometric and material nonlinearities are discussed.

1. Introduction

Arc-length techniques are numerical solution strategies that can successfully handle structural responses characterized by limit points. Pioneering work in this field is due to Riks [1] and Wempner [2], which introduced the idea of adding a load parameter, and consequently a constraint equation, as additional unknown of the problem. In these procedures, the load is free to increase or decrease throughout the iterative process, and the singularities of Jacobian matrix at the turning points are removed.

Further developments of the arc-length methods have been proposed in the early eighties. Ramm developed the updated normal path method [3], imposing the orthogonality between the iterative and the total increments. Crisfield [4] proposed a modified implementation of the arc-length solution scheme, focusing on the finite element implementation of the method and aiming to preserve the symmetric-banded nature of the equilibrium equations. This method is sometimes denoted as the spherical arc-length and the corresponding constraint equation is quadratic. A modified procedure was successively presented by the same author, based on the observation that material nonlinearities may determine convergence issues when standard arc-length procedures are adopted. To this aim, a line search algorithm was introduced in the solution process [5]. A comprehensive review of the arc-length methods is provided in [6]. In general, arc-length strategies are well suited to solve nonlinear elastic problems, but they are likely to suffer from convergence issues in presence of material softening [7,8]. In presence of delamination phenomena, the strain field is characterized by high localization in a restricted area surrounding the crack tip. It follows that the typical constraint equation based on global quantities tends to fail in capturing the process. To overcome this problem, the so-called local arc-lengths have been proposed by many authors [8–10]. The idea of these methods is to consider constraint equations based on the displacements of the dominant nodes, i.e. those nodes involved in the delamination process. In the implementation proposed by De Borst [11], the sliding displacement of a crack was adopted, while Rots and De Borst [12] considered the opening of the crack. May and Duan proposed the use of the relative displacement in the regions undergoing highly nonlinear processes [8]. In the work of Alfano and Crisfield [13], the control function is obtained as a weighted sum of a localized set of relative-displacement parameters.

In Ref. [14] a solution procedure is developed to study the delamination propagation based on the linear elastic fracture mechanics and Virtual Crack Propagation Technique (VCCT). The delamination length is used as the constraint variable, and the load increment size is controlled by means of the energy release rate. Despite the robustness of the local arc-lengths methods, the main drawback is that a priori knowledge of the failure process zone is needed, as the local degrees of freedom entering the constraint equation need to be identified. This restriction can be solved by adopting a path following constraint based on the energy release rate, as proposed by Gutierrez [15] and Verhoosel et al. [16]. This approach allows to develop robust methods, able to describe complex structural responses involving geometric and material nonlinearities. In any case, a double solution strategy is needed to switch from the elastic and the dissipative phases, as the dissipated

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energy is null during the elastic loading phases. A relevant aspect is then the criterion to handle the transition from the dissipative to the elastic solution procedure. In Refs. [15,16], the dissipative part is solved by considering a constraint based on the energy release rate, while the elastic phase is solved with a force control, activated on the basis of an energy threshold value. A first drawback of this strategy relies in the number of iterations, and the consequent impact on the efficiency of the method, characterizing the switch from the dissipative to the elastic solution procedure. A second restriction is the inability of the force-control to tackle responses characterized by purely elastic unloading phases, such as in the cases of snap-through or snap-back phenomena.

In this paper, a class of methods, hereinafter denoted as hybrid-methods, is presented. The methods are based on the combined use of geometric and dissipative constraint equations, which are updated at each step of the solution process on the basis of the damage state of the structure. Goal of these procedures is to guarantee robustness and improved computational efficiency, avoiding abrupt time-consuming transitions between the solution strategies for the dissipative and the elastic phase.

Preliminary aspects of the arc-length solution procedures are reviewed in Section 2, and the main features of the hybrid methods are presented in Section 3. The equations describing the hybrid-Riks method are derived in Section 3.1, while the Crisfield version is proposed in Section 3.2 together with a novel technique for the root selection of the quadratic constraint equation. The application of the hybrid-methods to three numerical examples is discussed in Section 4, where the performance is discussed in terms of robustness and computational efficiency.

2. Preliminaries on the arc-length methods

The finite element approximation of the nonlinear structural problem can be represented by the following set of discrete equilibrium equations:

$$\mathbf{f}_{int}(\mathbf{a}) = \mathbf{f}_{ext} \tag{1}$$

where the vectors $\mathbf{f}_{int}$ and $\mathbf{f}_{ext}$ denote the internal forces and the applied loads, respectively. The vector $\mathbf{a}$ collects the degrees of freedom of the problem and, for a displacement-based approach, they coincide with the nodal displacements. The set of equations of Eq. (1) is often solved with a force-control, meaning that the load is progressively increased, starting from the unloaded configuration, in the context of Newton-like iterations. In this case, unloading paths cannot be captured. This restriction can be avoided if a displacement control is conducted, consisting in specifying, step by step, the imposed value of the displacements at a given set of nodes. This approach can be applied to analyze a wide range of structural problems, but is not adequate to trace the full equilibrium path in those cases characterized by snap-backs.

To overcome the limitations related to the incremental solution procedures, the arc-length methods can be successfully adopted. These approaches are based on the representation of the external load as the product between a vector defining the shape of the load set, $\hat{\mathbf{f}}$, and a scalar parameter λ which determines the magnitude of the load. The load factor λ is then part of the problem unknowns, and its value is free to increase or decrease during the solution procedure. The presence of the additional unknown λ makes the system of Eq. (1) underdetermined, and one further equation, i.e. the constraint equation, is needed. The structural problem is then formulated as:

$$\begin{cases} \mathbf{f}_{int}(\mathbf{a}, \lambda) - \lambda\hat{\mathbf{f}} = 0 \\ g(\mathbf{a}, \lambda) = 0 \end{cases} \tag{2}$$

where the second term is a scalar equation defining the constraint.

The augmented system of Eq. (2) is usually solved by means of a predictor-corrector scheme. In the first step, the predictor provides an approximate solution. In the second step, the corrector phase, the predictor solution is used as the initial guess for an iterative procedure based on the Newton-Raphson method.

The problem unknowns of Eq. (2) are decomposed as:

$$\Delta\mathbf{a} = \Delta\mathbf{a}_j + d\mathbf{a}, \quad \Delta\lambda = \Delta\lambda_j + d\lambda \tag{3}$$

where $\Delta\mathbf{a}$ and $\Delta\lambda$ denote the total increments at the current step, $\Delta\mathbf{a}_j$ and $\Delta\lambda_j$ are the increments at the last iteration j, and $d\mathbf{a}$ and $d\lambda$ are the increments referred to the current iteration (corresponding to the index $j+1$). The index $j=0$ denotes the predictor solution, while the values $j = 1, \dots, N$ define the corrector iterations. A graphical representation of Eq. (3) is reported in Fig. 1, where the iterations of the Newton's method and the related increments of the unknowns are reported for a single load step.

By application of the Newton-Raphson method to Eq. (1), the following linearized set of equations is obtained:

$$\begin{cases} \mathbf{K}d\mathbf{a} - d\lambda\hat{\mathbf{f}} = \mathbf{r} \\ \mathbf{h}^T d\mathbf{a} + w d\lambda = -g_j \end{cases} \tag{4}$$

where:

$$\mathbf{K} = \frac{\partial \mathbf{f}_{int}}{\partial \mathbf{a}}, \quad \mathbf{h}^T = \frac{\partial g}{\partial \mathbf{a}}, \quad w = \frac{\partial g}{\partial \lambda}, \quad g_j = g(\Delta\mathbf{a}_j, \Delta\lambda_j) \tag{5}$$

and the residual $\mathbf{r}$ is defined as:

$$\mathbf{r} = \Delta\lambda_j\hat{\mathbf{f}} - \mathbf{f}_{int}(\Delta\mathbf{a}_j, \Delta\lambda_j) \tag{6}$$

The system of Eq. (4) can be solved by substitution, so that the iterative increments $d\mathbf{a}$ and $d\lambda$ are derived as:

$$\begin{aligned} d\mathbf{a} &= \mathbf{a}_I + d\lambda\mathbf{a}_{II} \\ d\lambda &= -\frac{g_j + \mathbf{h}^T\mathbf{a}_{II}}{\mathbf{h}^T\mathbf{a}_I + w} \end{aligned} \tag{7}$$

where:

$$\mathbf{a}_I = \mathbf{K}^{-1}\hat{\mathbf{f}} \quad \mathbf{a}_{II} = \mathbf{K}^{-1}\mathbf{r} \tag{8}$$

After substituting Eq. (7) into Eq. (3), the total increment of the current step can be determined.

Traditional arc-length methods – two examples are given by the Riks [1] and the Crisfield [4] formulations – are based on constraint equations that prescribe a geometric relation between the applied loads and the increments of the displacement vector. For the Riks method, the orthogonality is imposed between the iterative

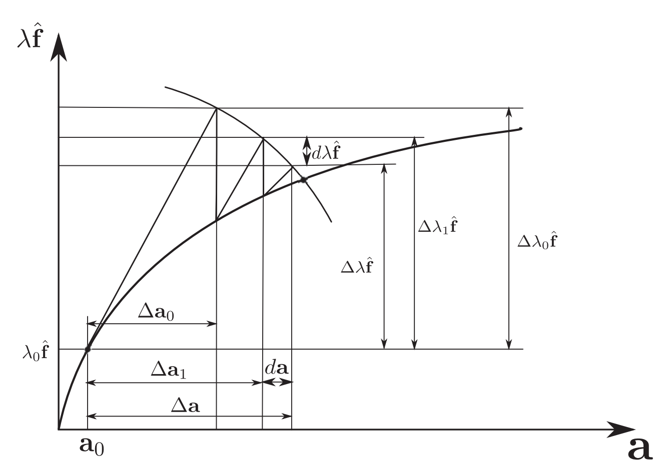

Fig. 1. Graphical representation of the decomposition of the unknowns.