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A new solution method for free vibration analysis of rectangular laminated composite plates with general stacking sequences and edge restraints

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ABSTRACT

A method is presented to study the free vibrations of rectangular laminated composite plates with general layups and arbitrary boundary conditions. Based on the first-order shear deformation theory, the governing differential equations and boundary conditions are deduced via Hamilton's principle. Generalised displacements are expanded as series with Legendre polynomials as the base functions. A generalised eigenvalue problem is obtained by following a variational approach, where energy functional is extremised and boundary conditions are introduced by means of Lagrange multipliers. In order to overcome some difficulties in obtaining the natural frequencies and corresponding mode shapes, a new numerical strategy is proposed.

1. Introduction

Laminated plates made of fibre-reinforced composite materials are used extensively in modern engineering – especially, in the aerospace, civil, mechanical, and nuclear industries – due to their excellent mechanical performances. Through proper arrangement of the stacking sequence (*i.e.*, the thickness, material properties, and fibre orientation of each constituent lamina), the strength and stiffness of a laminated plate can be tailored to satisfy the given design requirements. Nonetheless, when general (unsymmetric) stacking sequences are used, various coupling effects – such as bending-extension, bending-twisting, and extension-twisting couplings – may appear, because of the overall anisotropy of the laminated plate. Such effects alter the elastic behaviour and complicate the structural analysis with respect to homogeneous and isotropic plates [1].

On one hand, it is therefore very important to have a good understanding of the peculiarities of the mechanical behaviour of laminated composite plates. On the other hand, specialised, effective, and reliable tools are necessary for both the static and dynamic analyses of such composite structures. Various analytical solutions and numerical methods for the analysis of laminated composite plates are available in the literature. Recently, Sayyad and Ghugal [2] have presented a review article on the methods for the vibration analysis of laminated composite and sandwich plates. This problem has been studied intensively during the last decades. However, most studies are confined to special cases, such as cross-ply and angle-ply laminates, symmetric or anti-symmetric stacking sequences, and specific boundary conditions. Only a few studies concern laminated plates with general stacking sequences and elastic couplings, for which the governing differential equations are highly coupled and difficult to be solved. Moreover, as pointed out by many researchers, since composite laminates have very low transverse shear modulus compared to their in-plane elastic modulus, classical lamination theory may not be adequate even for the analysis of plates with high span-to-thickness ratios. Thus, shear deformation is another important aspect in the analysis of laminated composite plates.

A number of plate theories accounting for shear deformation have been proposed. In the simplest formulations, a constant transverse displacement in the thickness direction is considered and thus the transverse normal strain and stress are neglected. According to the literature, this assumption leads to accurate results only if the analysed plates have a sufficiently large side-to-thickness ratio. Instead, for lower values of the abovementioned ratio, the contribution of the transverse normal strain and stress should be considered to have more accurate results. Among the shear deformation theories in which the abovementioned strain

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and stress are not neglected, we recall the pioneering works by Lo et al. [3,4] and Kant and Owen [5,6]. Batra and Vidoli [7] have developed a three-dimensional mixed variational principle to derive a Kth-order two-dimensional linear theory for an anisotropic homogeneous piezoelectric plate. The mechanical and electric displacements, the electric potential, and the in-plane components of the stress tensor are expressed as a finite series of order K in the thickness coordinate. Besides, the transverse normal and shear stresses, and the transverse electric displacement are expressed as a finite series of order $(K+2)$ in the thickness coordinate. Carrera and Kroplin [8] have presented another way to consider the transverse normal strain and stress in the analysis of multilayered plates.

An exhaustive review of all of the higher-order shear deformation theories would however be too extensive and fall out of the scopes of the present paper. The interested reader can also refer to the theories presented by Librescu et al. [9], Reddy [10], and Fiedler et al. [11]. In what follows, we limit ourselves to cite a selection of papers that are directly related to the present work. Some of the cited theories will be used also for the comparison of results presented later on.

Aghababaei and Reddy [12] have reformulated the well-known shear deformation plate theory of Reddy [10] by using the nonlocal linear elasticity theory. They have analysed the bending and vibration behaviour of rectangular plates via Navier's method, which however can be used only for simply supported edges. Kant and Swaminathan [13], Robaldo et al. [14], Rao et al. [15], and Swaminathan and Patil [16] have carried out the free vibration analysis of laminated composite and sandwich plates based on shear deformation theories by using the finite element method (FEM). Also, Moita et al. [17] have investigated the buckling and free vibrations of laminated plates of arbitrary geometry and lay-ups using a discrete model based on single-layer higher-order shear deformation theory. Their model was implemented through an eight-node C^0 serendipity finite element with 10 degrees of freedom per node to contemplate general applications. Park et al. [18] have used the FEM to study the effects of skew angle and stacking sequence on the dynamic response of laminated skew plates. Free vibration analysis of symmetrically laminated, rectangular plates with clamped boundary conditions has been studied by using the hierarchical finite element method by Han and Petyt [19]. Rodrigues et al. [20] have combined the cell-based finite element method with the 4-noded quadrilateral mixed interpolation of tensorial components technique to study the static and dynamic response of laminated composite plates within Carrera's unified formulation (CUF) [21].

Lanhe et al. [22] have presented a novel numerical solution technique, called the moving least squares differential quadrature method, to study the free vibration problem of generally laminated plates based on first-order shear deformation theory. Recently, Ferreira et al. [23] have used CUF and the generalised differential quadrature technique to predict the static deformation and free vibration behaviour of thin and thick isotropic and cross-ply laminated plates. Ngo-Cong et al. [24] have proposed a one-dimensional integrated radial basis function collocation technique for the free vibration analysis of laminated composite plates using first-order shear deformation theory.

The Rayleigh–Ritz method is one of the most popular methods to obtain approximate solutions for the vibrational characteristics of laminated composite rectangular plates [25–27]. Baharlou and Leissa [28] have used this method for the analysis of vibration and buckling of generally laminated plates with various boundary conditions. Aydogdu and Timarci [29] have used the Rayleigh–Ritz method to carry out the free vibration analysis of cross-ply laminated square plates with different sets of boundary conditions. Hu et al. [30] have applied this method for the vibration of angle-ply laminated plates based on Mindlin's thick plate theory. Afterwards, Lee et al. [31] have employed the same method for the free vibration analysis of symmetrically laminated composite sandwich plates with elastic edge restraints. Khorshidi and Farhadi [32] have used the Rayleigh–Ritz method to investigate the hydrostatic vibrations of a laminated composite rectangular plate partially contacting with a bounded fluid.

In this paper, a new method is presented for the free vibration analysis of rectangular laminated composite plates with general stacking sequences. Fictitious elastic restraints are introduced on the plate's four edges, allowing any particular restraint condition to be simulated. With reference to the above discussion about shear deformation theories, it should be noted that the present formulation is based on first-order shear deformation theory, which is deemed sufficient to predict the vibrational characteristics of moderately thick plates. This choice is taken here for the sake of simplicity. However, the validity of such assumption will be assessed by comparing our results with those obtained by more accurate theories. Accordingly, the plate's displacement field is expressed in terms of the mid-plane displacements and rotations. The governing differential equations and boundary conditions are deduced via Hamilton's principle. Then, to determine the plate's natural frequencies and mode shapes, the generalised displacements are expanded as series with Legendre polynomials as the base functions. Thanks to the orthogonality of such polynomials, simple expressions are obtained for the plate's kinetic and strain energy. Subsequently, a generalised eigenvalue problem is obtained by following a variational approach, where an energy functional is extremised and the boundary conditions are introduced by means of Lagrange multipliers. The main advantage of using the method of Lagrange multipliers is that the assumed displacement functions do not have to satisfy *a priori* the boundary conditions of the problem. Lastly, several examples of plates with different stacking sequences and boundary conditions (clamped, simply supported, and free edges) are presented to demonstrate the effectiveness and accuracy of the method.

2. Formulation of the problem

2.1. Governing differential equations and boundary conditions

Let us consider a rectangular, laminated plate of uniform thickness h, length a, and width b, as shown in Fig. 1. A Cartesian reference system $Oxyz$ is fixed with the origin O at one of the plate's corners on the mid-plane, the x- and y-axes respectively aligned with the plate's longitudinal and transverse directions, and the z-axis completing the right-handed reference system. The laminate is made of many unidirectional fibre-reinforced laminae (or plies) stacked up in different orientations with respect to the reference axes.

Moving on to the boundary conditions, we assume that elastic restraints are present on the plate's contour. We denote with $k_d^{x=0}, k_d^{x=a}, k_d^{y=0}$, and $k_d^{y=b}$ the constants of the distributed elastic springs acting on the plate's four edges for each of the five local degrees of freedom $d \in \{u, v, w, \phi, \psi\}$. It should be noted that such general boundary conditions allow single degrees of freedom to be free or fixed, respectively by letting $k_d = 0$ or $k_d \to +\infty$. As a result, cases of clamped, simply supported, and free – as well as elastically restrained – edges can be considered by the method, as the examples in Section 3 will show. We denote with u, v, and w the mid-plane displacements in the x-, y-, and z-directions, respectively; ϕ and ψ are the angles of rotations of the normal to the mid-plane about the x- and y-axes, respectively.