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Tessellated continuum mechanics: A Galerkin finite element method

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ABSTRACT

This paper tests the hypothesis that the tessellation used in tessellated continuum mechanics can form a mesh in a continuous Galerkin finite element method. Although the tessellation is not unique, neither is it arbitrary, and its construction imposes constraints on any numerical analysis. A distinctive feature of the tessellation is that it can possess highly distorted elements yet—as a consequence of associated anisotropy in material properties—can still return accurate results.

The numerical procedure is tested on classical fractal porous geometries to demonstrate the potential of the method, and also illustrate the capability for analysis of disparate porous materials on continua.

1. Introduction

Porous materials are widespread in nature and can take on various forms; examples include biological tissue such as wood, cork and bone. Rocks and soils also often contain natural porous structures that may act as gas and oil reservoirs; as such, these structures are of particular interest to petrochemical industry, where there is currently much activity driving developments in *fracking*. Porous materials are also studied in diverse branches of modern engineering such as impact mechanics [1], fluid mechanics [2], poromechanics [3] and Printed Circuit Board (PCB) heat exchanger design [4]. This latter application exploits the porous structure to maximise the fluid contact area, and many porous heat exchangers embody this geometry through high thermal conductivity metallic foams such as copper and aluminium [5]. These are recognised to be an excellent choice for enhancing heat transfer, since they possess a large fluid-solid contact surface area, high thermal conductivities and provide good enhancement of fluid mixing [6]. Heat transfer provides a strong driver for the work described in this paper.

Although heat exchanger performance is enhanced by exploiting porous material, the heat transfer analysis itself faces serious difficulties arising principally from the complex geometry involved. The transport of heat and mass through porous media has received much attention for many decades in a wide variety of fields [7,8]; such approaches are indirect in that a continuum approach is retained, with the influence of the cellular structure relegated to coarse grained parameters such as permeability and porosity. Clearly such models are unable to capture any effects arising from refined changes in cellular designs. To incorporate more refined structures, Lattice Boltzmann Methods have been proposed as a possible way around the purely classical continuum description; Yan Su et al. [9] for example has performed comparison studies between direct and porous medium model heat exchangers. The coupling of discrete lattice models—which account for interactions between voids or particles via local potentials—with continuum models (e.g. quasi-continuum models [10]) is an area of active research [11]. The idea underpinning these types of approaches is the establishment of an appropriate continuum representation, where the material-constitutive response is informed by the lattice model. Although such approaches advance the analysis they are evidently restricted by the extraordinarily complex geometries involved, and are therefore unlikely to capture the complex flow and heat transfer physics in practical porous medium heat exchanger designs.

An alternative approach is to utilise the mathematics of fractal geometry [12] and/or involve extensions to traditional calculus by involving fractional derivatives or other such mathematical devices. Transport approaches involving fractals and fractional derivatives have been considered by Tarasov [13,14] and Ostoja-Starzewski [15,16]. Their approaches are untested and physically unrealisable, since they are founded upon transport forms that do not readily arise from the underpinning physics. More recently, a fractional differential equation has been investigated by Salvatore et al. [17] with the aim of establishing a more definitive connection between fractional calculus and fractal geometry. Similarly, Gianluca et al. [18] investigated fractal-porous materials founded on classical fractals, including the Cantor dust, Sierpinski carpet and the Sierpinski gasket using a fractional-order transport

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equation. It is transparent from the literature that most theories involving fractional derivatives have limited physical basis and offer little advance on classical parameter-continuum theory.

The approach adopted in this paper has at its foundation a classical control volume representation of the underlying physics. Transport equations in an integral (Euler) form are used in the analysis since integration—if correctly defined—readily caters for the discontinuous physics associated with fractals. The transport methodology is based on the existence of maps linking pre-fractals with the continuum which, in turn, is founded on the continuum hypothesis. These maps are named *hole-fill* maps due to the role they play in closing the pores in any fractal cellular representation to produce a tessellated continuum. We term the whole procedure of hole-fill, tessellated continua and subsequent analysis with transport equations as *tessellated continuum mechanics*.

Although the task of establishing hole-fill maps appears complex, a novel procedure introduced in Ref. [19] (and improved upon here) provides a straightforward route to obtain such constructions. The tessellated continuum—which forms an associated mesh in a Galerkin finite element analysis—is constructed in a process that essentially mirrors the fractal construction process itself. Thus, a fractal constructed through the iteration of n contraction maps also has n maps for the iterated construction of the tessellated continuum. The hole-fill maps can be constructed either by means of function composition or (more directly) by identifying the corresponding elements of a fractal with elements in the tessellated continua. The former is more suited for analytical work, with the latter suited to numerical analysis. For analytical work, the former approach is more suitable, since the requirement for the precise form of the governing partial differential equations can be elegantly established by means of the hole-fill map and the integral transport equations. For numerical analysis however, weighted forms of the transport equations can be established and immediately applied using the second approach, without recourse to partial differential equations. One of the novel contributions of this paper is to establish a finite element method on a tessellated continuum. We limit attention in this paper to a Galerkin formulation as this permits a commercial code to be readily employed. However, the appearance in the new approach of energy flux related terms at element edges—when coupled to the Galerkin formulation—limits the choice of fluid to those that are highly conducting and unable to support a temperature difference perpendicular to the direction of fluid flow. This does mean however that a modelling error [20,21] is present, which cannot be removed by means of mesh refinement. It is demonstrated in the paper that for the heat-exchanger problems considered that the modelling error is relatively small.

Other numerical methods (with or without elements [22–25]) could in principle be applied to solve the problems considered here. This could be done in one of two ways, i.e. by performing a numerical analysis on the tessellation (and mapping the results) or on the porous medium directly. No papers presently exist for analysis on a tessellation (as this paper is the first) but many methods have been applied to porous media [26–30], although these are limited to the solution of equations involving indirect representations of the geometry. There also exists a range of computational techniques for the efficient analysis of problems where the physical response is characterised by the size of the structure and the size of some underlying localised microscopic phenomena. The homogenisation approach [31] (which also capture the effect of geometry indirectly) can often be employed to analyse these types of problems and a computationally efficient approach can typically involve decomposition methods (see for example Refs. [32,33]).

This paper demonstrates an application of the Galerkin finite element method to a tessellated continuum to capture the energy transfers taking place on pre-fractal structures with applications to cellular heat exchangers. To achieve this, the general transport theory for pre-fractals and tessellated continua is presented in Section 2, where the physics of the two domains are related by the assumed existence of a hole-fill map. This is followed by the introduction to weighted transport equations in Section 3 and the establishment of the Galerkin finite element method for tessellated continua providing a convenient vehicle for numerical analysis of cellular designs. The whole procedure depends intimately on the tessellated continuum structure which is generated by means of a recursive method closely replicating the fractal generation process. Material properties for non-product fractals are considered in Section 4 along with corresponding tessellations. Of particular focus is the relationship between material properties on a tessellation and its relationship between corresponding pre-fractals. The tessellated finite element analysis method is introduced in Section 5 via some simple 1-D fractals; analytical 1-D solutions are obtained and contrasted against numerical predictions on different (but equivalent) tessellations. The process is repeated in Section 6 for product tessellations. In Section 7, thermal analysis is performed on non-product sets and predictions are contrasted with results obtained from the commercial package ABAQUS.

2. Tessellated continuum mechanics

The idea underpinning the tessellated approach is the assumed existence of a map (a *hole-fill* map) from a pre-fractal $\widehat{E}_k$ (which can be formed by the kth iterations of an Iteration Function Scheme (IFS) [34]) to a tessellation $\widehat{T}_k$ (which also can be formed in a similar fashion). Typical tessellations for some classical fractals are depicted in Figs. 1 and 2. The arrows depicted in the figures identify a corresponding selection of pre-fractal elements and tiles and it is important to appreciate that all pre-fractal elements have corresponding tiles. Physics taking place on $\widehat{E}_k$ is best represented in a weak sense using transport equations in integral form as these readily capture the highly discontinuous nature of the problem under consideration. Transport equations apply physical conservation laws to a control volume (identified here by Ω_s), within which the pre-fractal $\widehat{E}_k$ is at least in part embedded. A control volume is a continuous open set of points whose closure includes a continuous orientable boundary Γ_s. A typical transport equation for a stationary control volume takes the form

$$\frac{d}{dt}\int_{\Omega_s}\rho_s\psi_s dV_s+\int_{\Gamma_s}\rho_s\psi_s\underline{v}_s\cdot d\boldsymbol{\Gamma}_s=-\int_{\Gamma_s}\underline{J}_s\cdot d\boldsymbol{\Gamma}_s+\int_{\Omega_s}\rho_s b_s dV_s, \quad (1)$$

where ψ is a specific field variable, ρ is density, $\underline{v}$ is the material velocity, $\underline{n}$ is an outward pointing unit normal, $d\boldsymbol{\Gamma}=\underline{n}d\Gamma$, $\underline{J}\cdot\underline{n}$ is a flux, b is a source term and subscript s is used to indicate quantities in the physical space. It is assumed here that the fractal $\widehat{E}_k$ is not deforming although it is noted in passing that transport theory can readily account for such a scenario.

The pre-fractal $\widehat{E}_k$ is assumed to support mass and given that Ω_s can be made arbitrarily small it is possible that $\widehat{E}_k\cap\Omega_s=\Phi$, i.e. the intersection of the control volume and pre-fractal can under certain conditions be empty. This can happen in a situation where a control volume fits in a pore for example. Thus, in order to ensure the validity of Eq. (1) for arbitrary Ω_s some care is required particularly with flux terms; the absence of mass readily removes domain integrals but not necessarily the flux integral.

With the assumed existence of a tessellation $\widehat{T}_k$ it can be assumed further that there exists a similar transport equation for the stationary control volume Ω_r in which the tessellation $\widehat{T}_k$ is embedded. The transport equation for Ω_r is