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An efficient method for approximating resonance curves of weakly-damped nonlinear mechanical systems

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ABSTRACT

A method is presented for tracing the locus of a specific peak in the frequency response under variation of a parameter. It is applicable to periodic, steady-state vibrations of harmonically forced nonlinear mechanical systems. It operates in the frequency domain and its central idea is to assume a constant phase lag between forcing and response. The method is validated for a two-degree-of-freedom oscillator with cubic spring and a bladed disk with shroud contact. The method provides superior computational efficiency, but is limited to weakly-damped systems. Finally, the capability to reveal isolated solution branches is highlighted.

1. Introduction

The dynamical behavior of mechanical systems are governed by differential equations that are, in general, nonlinear in the describing coordinates. Once the nonlinear terms in the differential equations become relevant in the considered dynamical regime, we refer to these systems as nonlinear systems. Examples are systems with contact or dry friction, fluid–structure-interaction or large deflections. The nonlinear character makes their design and analysis more difficult and usually necessitates for appropriate numerical procedures. In the design process, it is often of interest to predict the vibration behavior in certain ranges of parameters. In the presence of sustained external forcing, the phenomenon of resonance is of particular concern. In this case, the steady-state vibrations can reach high levels, which may lead to structural damage and noise. To avoid resonances, or to ensure that the resonant vibration level is tolerable, it is important to predict resonance frequencies and associated vibration amplitudes. Throughout this work, only numerical methods are addressed, since the different analytical techniques are strictly limited in their range of applicability. Furthermore, we discuss only methods capable of analyzing the dependence on generic parameters. In contrast, methods limited to the analysis of only specific parameters are excluded from the discussion. An important example are nonlinear modal analysis methods [1], which are only capable of revealing the dependence of resonances on the excitation level.

The most straightforward way to obtain resonance frequencies and amplitudes is the computation of individual frequency responses for a discrete set of parameters. This technique often requires a large number of frequency response computations, because the system behavior can exhibit regimes of high or low sensitivities, and the associated parameter ranges are not a priori known. This might result in prohibitive computational effort.

An alternative technique was originally proposed by Petrov [2], and has been frequently applied since then, e.g. in [3]. The method aims at reducing the computational effort compared to crude forced response computations by directly determining the resonance curves, i.e., the curves that trace the locus of a specific peak of the frequency response under the variation of a parameter. To this end, a so-called resonance condition is introduced to the problem formulation, which requires the solution point to have a horizontal tangent in the amplitude–frequency plane. To this end, the unknown resonance frequency becomes part of the sought solution. The *horizontal-tangent method* is formulated in the frequency domain in the framework of the high-order harmonic balance method. The extended set of equations is solved using a gradient-based method in conjunction with a predictor–corrector arc-length continuation method. Since the computation of the tangent requires already the first-order derivatives, the gradient-based solution requires second-order derivatives of the residual with respect to the unknowns. This method can be categorized

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as a *first-order method*, in the sense that the formulation of the problem already involves first-order derivatives. Computing the required second-order derivatives can result in a huge computational burden, in particular if they are approximated by a finite difference scheme. But even in the case of analytical derivatives, the plain evaluation of their algebraic forms involves a considerable overhead. If this overhead is in the order of magnitude of the potential computational savings achieved by the direct analysis of resonances, this overhead can render this technique useless. Moreover, the analytical calculation of second-order derivatives can become an exhaustive and tedious task in the case of generic nonlinearities. For the class of piecewise polynomial systems, fortunately, automated frameworks are available [4].

In order to gain further computational savings in the computation of resonance curves, we propose a *zeroth-order method* in this work, i.e., a method that does not involve any derivatives in the formulation of the governing equations. The criterion is based on the phase lag between response and forcing. To this end, the notion of phase must be compatible with the analysis method, and we therefore place the method in the frequency domain framework.

In this paper we present this new method, which we refer to as phase-resonance method. We recap the harmonic balance method and present the additional phase lag criterion in Sections 2.1 and 2.2, respectively. In addition we show a qualitative analysis of the new method in comparison to the horizontal-tangent method in Section 2.3. In Sections 3 and 4, the method is validated for two nonlinear mechanical systems, namely a two degree-of-freedom (2DOF) oscillator with cubic spring, and a bladed disk with nonlinear shroud contact, respectively. Special attention is paid to the strongly nonlinear regime, where it is demonstrated that the method may also be useful to gain insight into the complicated behavior of dynamical systems.

2. A novel method for the direct computation of forced resonances of nonlinear systems

2.1. Recap of the harmonic balance method

The harmonic balance method[1] is a widely-used method to numerically compute the periodic, steady state oscillations of nonlinear dynamical systems. Consider a nonlinear mechanical system that has already been discretized in space and is described in terms of N_{DOF} generalized coordinates assembled in the vector $\mathbf{q}$. The motions $\mathbf{q}(t)$ of the system are governed by a set of ordinary differential equations,

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{C}\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)+\mathbf{f}_{\mathrm{nl}}(\mathbf{q}(t),\dot{\mathbf{q}}(t),\lambda)-\mathbf{f}_{\mathrm{ex}}(t,\lambda)=\mathbf{0}, \tag{1}$$

and appropriate initial conditions $\mathbf{q}(0)=\mathbf{q}_0$ and $\dot{\mathbf{q}}(0)=\mathbf{u}_0$. In Eq. (1) $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix, $\mathbf{f}_{\mathrm{ex}}(t,\lambda)$ is the vector of excitation forces, and $\mathbf{f}_{\mathrm{nl}}$ is the vector of nonlinear forces. Without loss of generality, it is assumed that $\mathbf{f}_{\mathrm{nl}}$ is essentially nonlinear in $\mathbf{q},\dot{\mathbf{q}}$ so that the linearized behavior is fully accounted for in the structural matrices $\mathbf{K}$ and $\mathbf{C}$, respectively. Furthermore t is the time and λ is a parameter which may influence the excitation and/or the nonlinear force. It should be noted that the approach can also be applied if the structural matrices are parameter dependent. This case is merely excluded from the further development for the sake of an easier presentation.

[1] In the literature, other widely used names for the method described here are the Describing Function method and the method of Krylov–Bogoliubov–Mitropolsky. Moreover, the prefix 'multi' or 'high-order' are often used for the harmonic balance method in order to clarify the difference to the single-term variant which only considers the fundamental harmonic.

In the harmonic balance method, periodic oscillations of the generalized coordinates $\mathbf{q}(t)$ with the fundamental frequency ω are assumed. The unknown coordinates are approximated by a truncated Fourier series with the harmonic order N_{h}:

$$\mathbf{q}(t)\approx\mathbf{Q}_0+\sum_{n=1}^{N_{\mathrm{h}}}\mathbf{Q}_n^{(c)}\cos(n\omega t)+\mathbf{Q}_n^{(s)}\sin(n\omega t). \tag{2}$$

The generalized velocities and accelerations are determined by deriving Eq. (2) with respect to time,

$$\dot{\mathbf{q}}(t)\approx\sum_{n=1}^{N_{\mathrm{h}}}-\mathbf{Q}_n^{(c)}n\omega\sin(n\omega t)+\mathbf{Q}_n^{(s)}n\omega\cos(n\omega t), \tag{3}$$

$$\ddot{\mathbf{q}}(t)\approx\sum_{n=1}^{N_{\mathrm{h}}}-\mathbf{Q}_n^{(c)}n^2\omega^2\cos(n\omega t)-\mathbf{Q}_n^{(s)}n^2\omega^2\sin(n\omega t). \tag{4}$$

For convenience, the Fourier coefficients are assembled in the following vectors:

$$\tilde{\mathbf{Q}}=\begin{pmatrix}(\mathbf{Q}_0)^{\mathrm{T}} & \left(\mathbf{Q}_1^{(c)}\right)^{\mathrm{T}} & \left(\mathbf{Q}_1^{(s)}\right)^{\mathrm{T}} & \dots & \left(\mathbf{Q}_{N_{\mathrm{h}}}^{(s)}\right)^{\mathrm{T}}\end{pmatrix}^{\mathrm{T}}. \tag{5}$$

Herein, $()^{\mathrm{T}}$ denotes the transpose. Substitution of Eqs. (2)–(4) into Eq. (1) generally produces an error term considering that only a limited number of harmonics is taken into account. It is then required that the original equation of motion is weakly enforced with respect to suitable weighting functions. Following the Galerkin idea, the base functions are utilized for this projection, which is known as Fourier–Galerkin projection [5], and results in the following set of nonlinear algebraic equations,

$$\mathbf{S}(\omega)\tilde{\mathbf{Q}}+\tilde{\mathbf{F}}_{\mathrm{nl}}\left(\tilde{\mathbf{Q}}\right)-\tilde{\mathbf{F}}_{\mathrm{ex}}=:\tilde{\mathbf{R}}\left(\tilde{\mathbf{Q}},\omega\right)\overset{!}{\rightarrow}0. \tag{6}$$

The Fourier coefficients of the forces, Herein, $\tilde{\mathbf{F}}_{\mathrm{nl}}$ and $\tilde{\mathbf{F}}_{\mathrm{ex}}$ take a form analogous to Eq. (5). In Eq. (6), $\tilde{\mathbf{R}}$ is the residual function, of which a zero characterizes a forced response approximation for a certain frequency ω. $\mathbf{S}(\omega)$ is the dynamic stiffness matrix and can be expressed as

$$\mathbf{S}(\omega)=\mathbf{D}^2(\omega)\otimes\mathbf{M}+\mathbf{D}^1(\omega)\otimes\mathbf{C}+\mathbf{D}^0(\omega)\otimes\mathbf{K}, \tag{7}$$

with the frequency-domain derivative matrix

$$\mathbf{D}(\omega)=\begin{pmatrix}0 & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & 1\cdot\omega & \dots & 0 & 0\\ 0 & -1\cdot\omega & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 0 & N_{\mathrm{h}}\cdot\omega\\ 0 & 0 & 0 & \dots & -N_{\mathrm{h}}\cdot\omega & 0\end{pmatrix}. \tag{8}$$

and $\otimes$ is the Kronecker product.

In Eq. (6), the coefficients of corresponding sine and cosine functions are thus enforced to add up to zero, that is they are 'balanced', hence the appropriate name 'harmonic balance method'. While $\tilde{\mathbf{F}}_{\mathrm{ex}}$ is usually given, the values of $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{F}}_{\mathrm{nl}}$ are unknown and are determined as the solution of Eq. (6). The solution is usually computed numerically and iteratively. The iterative scheme starts with a well-chosen vector $\tilde{\mathbf{Q}}_{(0)}$. A crucial task is the computation of the fourier coefficients $\tilde{\mathbf{F}}_{\mathrm{nl}}$ of the nonlinear forces. Closed-form expressions of the nonlinear forces in the frequency domain are only available in special cases. In the general case, the nonlinear forces can be more easily evaluated in the time domain. For this purpose, the Alternating Frequency/Time (AFT) method [6,7] is very popular. This method can be summarized as

$$\tilde{\mathbf{F}}_{\mathrm{nl}}=\mathrm{DFT}\left[\mathbf{f}_{\mathrm{nl}}\left(\mathrm{iDFT}\left[\tilde{\mathbf{Q}}\right]\right)\right], \tag{9}$$