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Finite element method on fractional visco-elastic frames

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ABSTRACT

In this study the Finite Element Method (FEM) on viscoelastic frames is presented. It is assumed that the Creep function of the constituent material is of power law type, as a consequence the local constitutive law is ruled by fractional operators. The Euler Bernoulli beam and the FEM for the frames are introduced. It is shown that the whole system is ruled by a set of coupled fractional differential equations. In quasi static setting the coupled fractional differential equations may be decomposed into a set of fractional viscoelastic Kelvin–Voigt units whose solution may be obtained in a very easy way.

1. Introduction

Finite Element Method (FEM) on viscoelastic structures is of relevant interest in many engineering problems. The FEM on frames is usually treated in literature by considering a local visco-elastic stress strain constitutive relation. Classical constitutive relation for visco-elastic materials employs combinations of purely elastic and purely viscous elements (Kelvin Voigt unit). In order to obtain a satisfactory solution for real viscoelastic material behavior, however, a large number of simple elements are required and hence a large number of materials parameters. Instead, use of fractional derivatives to describe viscoelastic materials leads, in a natural way, to model real viscoelastic behavior using very few parameters [1,2].

As a matter of fact, the order of differentiation of the strain characterizes the material behavior [3]. Having viscoelastic materials a non-integer value in the range [0,1], where the value zero characterizes a solid material and the value 1 a Newtonian fluid. Moreover, using this approach the material relaxation function follows a power-law decay instead of an exponential one, power-law observed for a wide range of engineering materials how it has been assessed by Nutting [4]. Based on this observation, starting from the second part of the last century, a lot of theoretical [5–13] and experimental researches [14–16] have been carried out confirming the Nutting experiences and assessing that the local constitutive viscoelastic law is ruled by fractional order operators. Starting from fractional viscoelastic constitutive laws the time dependent fractional differential equations of Eulero Bernoulli beam under transversal loads has been treated in recent literature [17,18]. On the other hand the FEM is a versatile method that allows us to solve more complex problems. This issue is addressed in a recent paper by Schmidt et al. [19].

This paper aims to give an insight on the quasi static analysis of frames, discretized in finite elements, with local fractional constitutive law and solving the fractional differential equations in a very simple way that may be easily implemented in computer codes.

Three main cases are treated that almost cover all the various cases of engineering interest: (i) the frame is constituted by a unique fractional viscoelastic material; (ii) the viscoelastic frame is infilled with fractional viscoelastic devices, characterized by different power-law in the creep, to mitigate the total response; and (iii) the frame is composed by two different viscoelastic elements, such an example steel elements and concrete elements (seismic walls), or base isolated frames with viscoelastic rubber.

It is shown that in all these cases the coupled fractional differential equations in quasi static setting may be decoupled into a set of fractional Kelvin Voigt elements whose solution may be obtained in a very easy way in terms of Mittag Leffler series or by using other available techniques to integrate single fractional differential equations [20].

2. Preliminary concepts and definitions

The theory of viscoelastic hereditary materials is based upon the knowledge of the so called *Creep function*, denoted as $D(t)$, that is the strain history due to an unitary stress. By using the Boltzmann superposition principle the strain history $\varepsilon(t)$ due to the stress history $\sigma(t)$ may be written in the form

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$$\varepsilon(t) = D(t)\sigma_0 + \int_0^t D(t-\tau)\dot{\sigma}(\tau)d\tau \tag{1}$$

where σ_0 is the stress at $t = 0$. An inverse relationship of Eq. (1) may be assessed by means of the so called *relaxation function*, denoted as $E(t)$ that is the stress history due to an imposed unitary strain. Using again the Boltzmann superposition principle we get

$$\sigma(t) = E(t)\varepsilon_0 + \int_0^t E(t-\tau)\dot{\varepsilon}(\tau)d\tau \tag{2}$$

where ε_0 is the (assigned) strain at $t = 0$. As soon as we assume for $E(t)$ a power law decaying function

$$E(t) = \frac{E_\beta}{\Gamma(1-\beta)}t^{-\beta} \tag{3}$$

being E_β, β the relevant parameters that characterize the material at hand (obtained by best fitting based upon experimental data) and $\Gamma(\cdot)$ is the Euler Gamma function then Eq. (2) gives

$$\sigma(t) = E(t)\varepsilon_0 + E(\beta)\left({}^C D^\beta_{0+}\varepsilon(t)\right) \tag{4}$$

where the symbol $({}^C D^\beta_{0+}\varepsilon(t))$ is the so-called Caputo's fractional derivative of order β, that is

$$\left({}^C D^\beta_{0+}\varepsilon(t)\right) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-\tau)^{-\beta}\dot{\varepsilon}(\tau)d\tau \tag{5}$$

Without loss of generality in the following we suppose that the system is quiescent at $t = 0$ (i.e. $\sigma_0 = 0,\ \varepsilon_0 = 0$), in this way in Laplace domain creep and relaxation are related each other by the fundamental relation

$$D(s)E(s) = \frac{1}{s^2} \tag{6}$$

being $D(s)$ and $E(s)$ the Laplace transform of $D(t)$ and $E(t)$, respectively.

Computing Laplace transform of the relaxation function, (3), after some simple algebra we get

$$D(t) = \frac{E_\beta^{-1}}{\Gamma(1+\beta)}t^\beta \tag{7}$$

by inserting this expression in Eq. (1) and integrating by parts we get

$$\varepsilon(t) = \frac{1}{E_\beta}\left(I^\beta_{0^+}\sigma(t)\right) \tag{8}$$

where the symbol $\left(I^\beta_{0^+}\sigma(t)\right)$ is the so-called Riemann Liouville fractional integral defined as

$$\left(I^\beta_{0^+}\sigma(t)\right) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\sigma(\tau)d\tau \tag{9}$$

It is worth noticing that for $\beta = 0$, since $({}^C D^\beta_{0+}\varepsilon(t)) \equiv \varepsilon$, then the purely elastic behavior is restored. Moreover for $\beta = 1$, since $({}^C D^\beta_{0+}\varepsilon(t)) \equiv \dot{\varepsilon}$, the behavior of purely Newtonian viscous fluid is restored. For any other value of $\beta:\ 0 < \beta < 1$ a variety of fractional viscoelastic behavior is evidenced. A wide discussion on this subject may be found in [21].

It is apparent that in the case in which the relaxation function is that given in Eq. (3), as we assume a constant hydrostatic pressure on the solid, as $t \to \infty$ the volume will be zero and this is in contrast with the experimental evidence. In order to avoid this problem usually the relaxation function is assumed in the form

$$E(t) = E_0 + \frac{E_\beta}{\Gamma(1-\beta)}t^{-\beta} \tag{10}$$

in this way as $t \to \infty$ the elementary volume under hydrostatic pressure will remain finite. In this case Eq. (4) is enriched by one purely elastic term that is

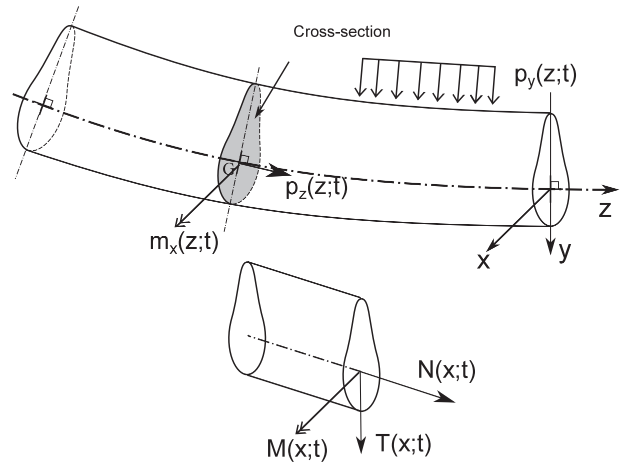

Fig. 1. Euler–Bernoulli beam reference system.

$$\sigma(t) = E_0\varepsilon(t) + E(\beta)\left({}^C D^\beta_{0+}\varepsilon(t)\right) \tag{11}$$

The inverse relation of Eq. (10) in Laplace domain is not so elementary as in the case of $E_0 = 0$, since $D(t)$ involves Mittag–Leffler expansion and then a series of Riemann Liouville integrals appears. This issue will be addressed later on.

Once the local constitutive laws has been assumed in Eq. (11), then the Euler–Bernoulli beam with fractional constitutive law may be derived.

Let the beam in Fig. 1, be referred to an anti-clockwise axes and let z be the axis joining the centroids of the various transverse sections. Let x and y be the principal axes of the transverse sections. Moreover let us suppose that the external loads per unit length lie in the plane $y-z$ denoted as $p_y(z,t)$ (transversal) $p_z(z,t)$ (axial). Moreover let $m(z,t)$ be the external moment per unit length, and let us denote $u_y(z,t)$ the deflection of the fibers in the plane xy. The elongation of the longitudinal fiber of length dz, is given as $dz\varepsilon(x,y,z;t)$ and in Euler–Bernoulli hypothesis we write:

$$\begin{aligned}\varepsilon_z(x,y,z;t) = \varepsilon_z(y,z;t) &= \frac{\partial u_z(z,t)}{\partial z} + y\frac{\partial \phi_x(z,t)}{\partial z} \\ &= \frac{\partial u_z(z,t)}{\partial z} - y\frac{\partial^2 u_y(z,t)}{\partial z^2}\end{aligned} \tag{12}$$

where $\phi_x(z,t) = -\frac{\partial u_y(z,t)}{\partial z}$ is the rotation angle of the cross section.

By inserting Eq. (11) follows that the stress σ_z, for a beam at rest at $t = 0$ is given as

$$\begin{aligned}\sigma_z(x,y,z;t) = \sigma_z(y,z;t) &= E_0\varepsilon_z(t) + E_\beta\left({}^C D^\beta_{0+}\varepsilon_z(t)\right) \\ &= E_0\frac{\partial u_z}{\partial z} + E_\beta\left({}^C D^\beta_{0+}\frac{\partial u_z}{\partial z}(t)\right) - E_0 y\frac{\partial^2 u_y}{\partial z^2} \\ &\quad - E_\beta y\left({}^C D^\beta_{0+}\frac{\partial^2 u_y}{\partial z^2}(t)\right)\end{aligned} \tag{13}$$

The total axial force $N(z;t)$ is then given as

$$N(z;t) = \int_A \sigma_z(y,z;t)dA = A\left(E_0\frac{\partial u_z}{\partial z} + E_\beta\left({}^C D^\beta_{0+}\frac{\partial u_z}{\partial z}(t)\right)\right) \tag{14}$$

The bending moment $M(z;t)$ is given as

$$\begin{aligned}M(z;t) &= -\int_A y\sigma_z(y,z;t)dA \\ &= -I_x\left(E_0\frac{\partial^2 u_y}{\partial z^2} + E_\beta\left({}^C D^\beta_{0+}\frac{\partial^2 u_y}{\partial z^2}(t)\right)\right)\end{aligned} \tag{15}$$