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Free vibration analysis using the transfer-matrix method on a tapered beam

Jung Woo Lee, Jung Youn Lee *

Dept. of Mechanical System Engineering, Kyonggi University, San 94-6, Eui-dong, Yeongtong-gu, Suwon-si, Gyeonggi-do 443-760, Republic of Korea

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ABSTRACT

A transfer-matrix method is developed to determine more accurate solutions to the free vibration characteristics of a tapered Bernoulli–Euler beam. The roots of the differential equation are determined by using the Frobenius method to obtain the power series solution for bending vibrations. This study examines the effect of various taper ratios on the eigenpairs of these beams, in which the height of the cross-section along the length is linearly reduced. In addition, the number of terms in the power series is investigated in detail for each taper ratio because the required number of terms depends upon the taper ratio.

1. Introductions

The analysis of bending vibrations for non-uniform beams (i.e., stepped or tapered beams) has been performed by a number of researchers using various approaches, such as the finite element, transfer-matrix, and other approximate methods [1–25]. The use of non-uniform beams to analyze the free vibration characteristics has been widely applied to engineering applications. Generally, a non-uniform beam is regarded as a tapered beam, which can be modeled by an idealized stepped uniform beam [23] and the appropriate shape function [1–16,18,21,22]. However, the vibration analysis for a tapered beam generally approximates the shape function because this approach can produce more accurate results and can simplify the computation. In addition, most studies assume that the cross-sectional dimensions along the length of a beam are linearly reduced [2,4,5,10,15,16,21,22,24]. In this regard, when the cross-sectional dimensions are assumed to be small in comparison with the total length of the beam element, the beam analysis can be simplified to a Bernoulli–Euler problem by ignoring the rotary inertia and shear deformation [6,8,13].

A dynamic stiffness method was implemented by Banerjee et al. [3], Leung and Zhou [5], Spyrakos and Chen [12] and Banerjee [23] to obtain the natural frequencies and mode shapes for such beam analysis problems. The method used by Banerjee et al. [3] involved a power series solution and used the Frobenius method to compute the roots of the corresponding differential equation. The natural frequencies and mode shapes, which were determined using the finite element method, are shown to provide excellent series convergence for tapered beams when considering the centrifugal force. The Rayleigh–Ritz method was used by Zhou and Cheung [1,6], Zhou [7] and Lu et al. [17] to solve for the bending vibrations of tapered beams, plates and multiple-stepped composite beams, respectively. The differential transform method was used by Ozgumus and Kaya [18] and Rajasekaran [14], and the transfer-matrix method was used by Stafford and Giurgiutiu [19], Attar [20], and Takahashi [25]. Gunda et al. [22] studied the effects of the centrifugal force and the taper ratio on the free vibration characteristics of high speed rotating beams using hybrid stiff-string-polynomial basis functions, and the results obtained using a new finite element approach have been shown to provide excellent series convergence.

For the problems described above, some researchers obtained the roots of the differential equation by using the power series solution obtained by the Frobenius method [3,10,15,22,23], and the more accurate roots have been computed successfully. Therefore, in this study, the Frobenius method is used to compute the roots of the differential equation, and the transfer-matrix method is used to determine the natural frequencies and mode shapes for the bending vibrations of various tapered beams. The transfer-matrix method can be used as a powerful numerical tool to investigate the vibration characteristics of beam elements. However, the use of the transfer-matrix method, which is used in the proposed method, has not been studied even though one can obtain the infinite natural frequencies and mode shapes using a single element for the bending vibrations of various tapered beams.

* Corresponding author. Tel.: +82 31 249 9811; fax: +82 31 244 6300.
E-mail address: jylee@kgu.ac.kr (J.Y. Lee).

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The objective of this study is to develop a transfer-matrix method that can be used to calculate more accurate solutions to the bending vibration characteristics of Bernoulli–Euler beams with various taper ratios and, subsequently, to determine the eigenpairs for a tapered beam in which the height of the cross-section of the rectangular beam is linearly reduced along its length. By using the differential equation, shear force and bending moment, which are deduced by different variational principles, the transfer-matrix method is formulated. The proposed method does not separate the mass and stiffness matrices, but this method has the same essential and important features of the conventional transfer matrix. Moreover, one of the more advantageous features of the present theory is that the use of a single element can produce infinite natural frequencies and mode shapes. However, the proposed method uses a truncated polynomial series, such as the Frobenius method, and depends on the number of terms in the power series for determining the more accurate roots of the differential equation. When expanding the number of terms in the power series to infinity, the roots of the differential equation become exact. However, to produce fast and efficient results, the number of terms in the power series needs to be investigated. Numerical results for validating the accuracy of the proposed method are compared with those presented in [3], and the number of terms in the power series required to increase the efficiency of the computation is investigated by a parametric study with respect to the taper ratios.

2. Theory

For a tapered Bernoulli–Euler beam, which is examined in this study, the total beam length is assumed to be significantly larger than its cross-sectional dimensions. Therefore, the shear deformation and rotary inertia can both be ignored, and only the bending vibration needs to be investigated. The notation and coordinate systems shown in Fig. 1 are used to solve for the free vibration

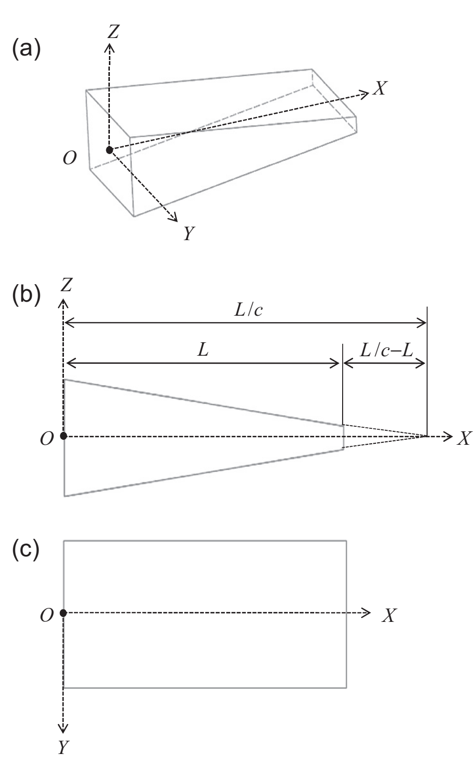

Fig. 1. Notation and coordinate system used for a tapered Bernoulli–Euler beam: (a) geometry of the tapered beam, (b) side view, and (c) top view.

characteristics of a tapered Bernoulli–Euler beam, where XYZ is the global coordinate system, L is the total length of the beam element, and c is the taper ratio.

The governing differential equation, shear force, and bending moment for the bending vibration of a tapered Bernoulli–Euler beam can be deduced by different variational principles, and the strain (U) and kinetic (T) energies are expressed as follows [6]:

$$U = \frac{1}{2}\int_0^L EI(x)(w'')^2 dx \tag{1}$$

and

$$T = \frac{1}{2}\int_0^L m(x)(\dot{w})^2 dx \tag{2}$$

where $EI(x)$, $m(x)$ and $I(x)$ are the variation of the bending stiffness, the mass per unit length according to the taper ratio and the geometric moments of inertia for the beam cross-section according to the taper ratio, respectively, and E is the elastic modulus of the beam. In addition, $w(= w(x,t))$ is the in-plane bending displacement. The prime and the dot symbols denote differentiation with respect to the distance x and time t, respectively. The variation of the bending stiffness and the mass for a tapered beam can be expressed as follows [8,10]:

$$EI(x) = EI_0\left(1 - c\frac{x}{L}\right)^3 \tag{3}$$

and

$$m(x) = m_0\left(1 - c\frac{x}{L}\right) \tag{4}$$

where EI_0 and m_0 are the bending stiffness and mass per unit length, respectively, for a uniform beam ($c = 0$).

By the variational principle, the differential equation can be expressed as follows [21]:

$$(EI(x)w'')'' + m(x)\ddot{w} = 0 \tag{5}$$

In addition, the shear force and bending moment can be defined by the following equations:

$$V(x,t) = -(EI(x)w'')' \tag{6}$$

and

$$M(x,t) = EI(x)w'' \tag{7}$$

where $V(x,t)$ is the shear force, and $M(x,t)$ is the bending moment.

Assuming harmonic vibrations with angular frequency (ω),

$$w(x,t) = W(x)\cos\omega t \tag{8}$$

where $W(x)$ represents the amplitudes of $w(x,t)$.

By substituting Eq. (8) into Eq. (5), the equation can be simplified as follows:

$$EI(x)W'''' + 2EI'(x)W''' + EI''(x)W'' - m(x)\omega^2 W = 0 \tag{9}$$

where $W = W(x)$.

By substituting Eqs. (3) and (4) into Eq. (9), the differential equation can be simplified in a nondimensional form as follows:

$$(1-\zeta)^3 W'''' - 6(1-\zeta)^2 W''' + 6(1-\zeta)W'' + \bar{\omega}^2(1-\zeta)W = 0 \tag{10}$$

where

$$\zeta = c\bar{x}, \quad \bar{\omega}^2 = -\frac{m_0\omega^2 L^4}{EI_0 c^4} \tag{11}$$

In addition, $W = W(\zeta)$, $\bar{\omega}$ is the nondimensional angular frequency, $\bar{x}(= x/L)$ is the nondimensional coordinate, and the parameter for the distance is x.

Eq. (10) can be simplified as follows: