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Analysis of no-tension structures under monotonic loading through an energy-based method

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ABSTRACT

An approach is proposed to estimate the collapse load of linear elastic isotropic no-tension 2D solids. The material is replaced by a suitable equivalent orthotropic material with spatially varying local properties. A non-incremental energy-based algorithm is implemented to define the distribution and the orientation of the equivalent material, minimizing the potential energy so as to achieve a compression-only state of stress. The algorithm is embedded within a numerical procedure that evaluates the collapse mechanisms of structural elements under monotonic loading. The accuracy of the method is assessed through comparisons with the "exact" results predicted by limit analysis.

1. Introduction

The safety assessment of the architectural heritage is nowadays an issue of paramount importance, both because of the need of avoiding to put human lives at risk, and because of the economic impact of this priceless heritage, for instance, on tourism industry.

In most countries, the architectural heritage consists of masonry buildings. Masonry, whether of stone or brick, is well known to be a material with low tensile strength. The inability of transferring significant tensile stresses is the reason for the extensive crack patterns that can be observed in many ancient buildings. The presence of cracks is not necessarily symptomatic of a possible collapse, as stresses can spontaneously attain a purely compressive state, which makes cracked regions unnecessary to the stability of the building.

Many formulations have been proposed in the last decades to analyze masonry structures, at different degrees of accuracy. Inelastic constitutive models, taking plastic strains and/or damage effects into account, were presented e.g. in [16,19,22], only to quote a few.

Several authors have proposed to analyze masonry structures using no-tension material models: this is why no-tension materials are sometimes referred to as "masonry-like" materials in the literature [2,10,13]. Neglecting the low tensile strength of masonry completely is a simplification that is far from being original: in the sixties of last century, Heyman [15] proposed to specialize limit analysis to masonry structures assuming the compressive strength to be unlimited and the tensile strength to vanish. This schematization is spontaneous for stone masonry, consisting of blocks separated by weak joints, but was adopted by other authors also for brick masonry [2,4,5].

Recently, Angelillo et al. [2] proposed an approach to analyze two-dimensional no-tension bodies subjected to given loads based on the unconstrained minimization of the potential energy with respect to the unknown displacement field. The numerical difficulties related to the enforcement of the no-tension constraint [10] are avoided by giving the strain energy density of the material different expressions, according to the sign of the principal stresses.

This idea was later exploited by Bruggi [9] to re-formulate the analysis of no-tension solids as a topology optimization problem. The equilibrium of a two-dimensional no-tension body is found searching for the distribution of an "equivalent" orthotropic material that minimizes the potential energy of the solid. The orthotropic material exhibits negligible stiffness in any direction along which a tensile principal stress must be prevented in the isotropic medium. Similarly to [2], this approach obtains the solution under given loads through a one-shot energy-based optimization, provided that the applied loads are compatible with the no-tension constraint.

In this work, an approach is presented to predict the carrying capacity of no-tension 2D structural elements, starting from that proposed in [9]. A numerical scheme is formulated to estimate the collapse load (and the relevant failure mechanism) of no-tension structures with a number of load steps much lower than that required by conventional incremental analyses.

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The layout of the paper is as follows. In Section 2 the mathematical problem that allows the no-tension isotropic solid to be analyzed as an equivalent orthotropic medium is briefly recalled. The distribution and the orientation of the orthotropic material are determined by minimizing the elastic strain energy of the solid. In Section 3, the model is implemented in an algorithm that allows the stress analysis of any 2D no-tension body to be directly carried out without the need of any incremental procedure (Section 3.1). Upon discretization of the problem, the constrained minimization of the objective function is carried out through mathematical programming [23]. If required, the entire load–displacement curve under monotonic loading can be derived at any degree of accuracy by running a sequence of independent analyses. A few comments on the proposed algorithm and comparisons with alternative optimization tools available in the literature are made in Section 3.2. The possibility of estimating the collapse load of the structure using an expressly developed algorithm is shown in Section 3.3. The effectiveness of the model in predicting the collapse load of various no-tension structures is assessed in Section 4. A preliminary discussion is made on the accuracy of the results obtained checking the no-tension condition only at the centroid of any finite element, or at each of the Gauss points of the element. Comparisons between the numerically estimated collapse loads and the values obtained using limit analysis specialized to no-tension materials are also presented. Finally, in Section 5 the main findings of the work are summarized and future extensions of the research are outlined.

2. Problem formulation

According to Bruggi [9], the equilibrium of any 2D isotropic linear elastic no-tension solid can be stated in weak form as follows:

$$\begin{cases} \min\limits_{\rho_1,\rho_2} & \frac{1}{2}\int_\Omega \mathbf{D}(\rho_1,\rho_2,\theta)\underline{\varepsilon}(\underline{u})\underline{\varepsilon}(\underline{u})d\Omega \\ \text{s.t.} & \int_\Omega \mathbf{D}(\rho_1,\rho_2,\theta)\underline{\varepsilon}(\underline{u})\underline{\varepsilon}(\underline{v})d\Omega = \int_{\Gamma_t}\underline{t}_0\cdot\underline{v}d\Gamma \text{ and } \underline{u}|_{\Gamma_u} = \underline{u}_0 \quad \forall \underline{v}, \\ & \theta \mid \tilde{z}_1 = z_I \text{ and } \tilde{z}_2 = z_{II}, \\ & \rho_1,\rho_2 \mid \sigma_I \le 0 \text{ and } \sigma_{II} \le 0, \\ & 0 < \rho_{min} \leqslant \rho_1,\rho_2 \leqslant 1. \end{cases} \tag{1}$$

In Eq. (1), Ω is the domain occupied by the solid, Γ_t is its free boundary and Γ_u its fixed boundary. $\underline{t}_0$ are prescribed tractions on Γ_t and $\underline{u}_0$ are prescribed displacements on Γ_u; body forces are neglected. $\underline{u}$ is the displacement field and $\underline{\varepsilon} = [\varepsilon_{11}\,\varepsilon_{22}\,2\varepsilon_{12}]$ is the array of the strain components in a global Cartesian reference system Oz_1z_2. The real isotropic no-tension material is replaced by an equivalent orthotropic material, with symmetry axes $\tilde{z}_1$ and $\tilde{z}_2$, coinciding with the principal stress directions, z_I and z_{II}, at any point of the real solid. θ is the angle between z_I and z_1. Indeed, from the optimal design of anisotropic elastic solids (see e.g. [18,20,21]), it is well known that the symmetry axes characterizing a maximum (or a minimum) stiffness layout are aligned to the principal stress directions. Unlike the approach followed in [9], where this alignment was iteratively enforced within the solution procedure, here the orientation field $\theta(z_1,z_2)$ is added to the unknown variables, and the expected alignment of symmetry axes and principal stress directions spontaneously arises from the minimization procedure.

The stress–strain law for the equivalent material is written as $\underline{\sigma} = \mathbf{D}\underline{\varepsilon}$, where $\underline{\sigma} = [\sigma_{11}\,\sigma_{22}\,\sigma_{12}]$ and $\mathbf{D}$ can be expressed as

$$\mathbf{D} = \mathbf{T}(\theta)\widetilde{\mathbf{D}}\mathbf{T}(\theta)^T, \tag{2}$$

being $\widetilde{\mathbf{D}}$ the elasticity matrix in the reference system $O\tilde{z}_1\tilde{z}_2$ and $\mathbf{T}$ a transformation matrix, depending on the orientation $\theta(z_1,z_2)$. Assuming plane stress conditions, $\widetilde{\mathbf{D}}$ reads

$$\widetilde{\mathbf{D}} = \frac{1}{1-\tilde{\nu}_{12}\tilde{\nu}_{21}}\begin{bmatrix} \widetilde{E}_1 & \tilde{\nu}_{12}\widetilde{E}_2 & 0 \\ \tilde{\nu}_{21}\widetilde{E}_1 & \widetilde{E}_2 & 0 \\ 0 & 0 & \widetilde{G}_{12}(1-\tilde{\nu}_{12}\tilde{\nu}_{21}) \end{bmatrix}, \tag{3}$$

where $\widetilde{E}_1$, $\widetilde{E}_2$ are the Young's moduli of the equivalent material along $\tilde{z}_1$ and $\tilde{z}_2$, respectively, $\widetilde{G}_{12}$ is the in-plane shear modulus and $\tilde{\nu}_{12}$, $\tilde{\nu}_{21}$ are Poisson's ratios. The equality $\tilde{\nu}_{12}\widetilde{E}_2 = \tilde{\nu}_{21}\widetilde{E}_1$ holds. $\mathbf{T}$ can be written as

$$\mathbf{T} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & 2\cos\theta\sin\theta \\ \cos\theta\sin\theta & -\cos\theta\sin\theta & \cos^2\theta-\sin^2\theta \end{bmatrix}. \tag{4}$$

The nondimensional "material densities" ρ_1 and ρ_2 ($\in (0,1]$) define the decrease of the elastic properties of the equivalent material with respect to those of the real material along $\tilde{z}_1$ and $\tilde{z}_2$ if one or both principal stresses are tensile, according to a generalization of the so-called SIMP material model (see e.g. [6,7]):

$$\begin{aligned} &\widetilde{E}_1 = \rho_1^pE, \quad \widetilde{E}_2 = \rho_2^pE, \quad \widetilde{G}_{12} = \rho_1^p\rho_2^p\frac{E}{2(1+\nu)}, \\ &\tilde{\nu}_{12} = \rho_2^p\nu, \quad \tilde{\nu}_{21} = \rho_1^p\nu. \end{aligned} \tag{5}$$

In Eq. (5), E and ν are the Young's modulus and the Poisson's ratio of the real isotropic material, respectively, and p is a penalization parameter (usually taken equal to 3). The normalized densities are given a strictly positive lower bound, ρ_{min}, to avoid any singularity in the stiffness matrix of the body, $\mathbf{K}$, when a finite element discretization is adopted.

3. Finite element analysis of no-tension structures

3.1. Direct analysis for any prescribed compatible load

The discretized form of the minimization problem (1) is implemented into a finite element program through a procedure called Solve, which allows the analysis of a linear elastic no-tension structure to be directly carried out for any prescribed load compatible with the mechanical behavior of the material.

Algorithm 1.

procedure Solve
  $j = 0, \quad \psi(0) = 0, \quad \Delta\psi = \infty$
  Initialize variables: $x_{1e} = x_{2e} = 0.5, \; t_e = \pi/2, \quad \forall e$
  while $\Delta\psi > \Delta\psi_{tol}$ **do**
    $j = j+1$
    Solve $\mathbf{K}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{t})\mathbf{U} = \mathbf{F}$
    Compute $\psi(j) = \frac{1}{2}\mathbf{U}^T\mathbf{K}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{t})\mathbf{U}$
    Evaluate $\Delta\psi = |\psi(j) - \psi(j-1)|$
    Assign $\hat{x}_{ie}$, for $i = 1,2$, such that:
    $\begin{cases} \hat{x}_{ie} = x_{ie}, & \text{if } \sigma_{e,i} \le 0, \\ \hat{x}_{ie} = kx_{ie}, & \text{otherwise.} \end{cases}$
    Compute $\hat{\psi} = \frac{1}{2}\mathbf{U}^T\mathbf{K}(\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2,\mathbf{t})\mathbf{U}$
    Compute reduced sensitivities $\partial\hat{\psi}$
    $\begin{cases} \frac{\partial\hat{\psi}}{\partial x_{ie}} = \frac{1}{2}\mathbf{U}_e^T\frac{\partial}{\partial x_{ie}}\mathbf{K}_e(\hat{x}_{1e},\hat{x}_{2e},t_e)\mathbf{U}_e, & \text{for } i = 1,2 \\ \frac{\partial\hat{\psi}}{\partial t_e} = \frac{1}{2}\mathbf{U}_e^T\frac{\partial}{\partial t_e}\mathbf{K}_e(\hat{x}_{1e},\hat{x}_{2e},t_e)\mathbf{U}_e \end{cases}$
    Run MMA$(\hat{\psi},\partial\hat{\psi})$ to update x_{1e}, x_{2e}, t_e
  end while
  Solve $\mathbf{K}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{t})\mathbf{U} = \mathbf{F}$
end procedure