Computers and Structures 151 (2015) 11–19

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

The method of fundamental solutions for solving direct and inverse Signorini problems

A. Karageorghis [a,*], D. Lesnic [b], L. Marin [c,d]

[a] Department of Mathematics and Statistics, University of Cyprus/Πανεπιστήμιο Κύπρου, P.O. Box 20537, 1678 Nicosia/Λευκωσία, Cyprus
[b] Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK
[c] Department of Mathematics, Faculty of Mathematics and Computer Science, University of Bucharest, 14 Academiei, 010014 Bucharest, Romania
[d] Institute of Solid Mechanics, Romanian Academy, 15 Constantin Mille, P.O. Box 1-863, 010141 Bucharest, Romania

ARTICLE INFO

Article history:
Received 15 October 2014
Accepted 4 January 2015

Keywords:
Signorini problem
Inverse problem
Method of fundamental solutions
Nonlinear optimization

ABSTRACT

Signorini problems model phenomena in which a known or unknown portion of the boundary is subjected to alternating Dirichlet and Neumann boundary conditions. In this paper, we apply the method of fundamental solutions (MFS) for the solution of two-dimensional both direct and inverse Signorini problems for the Laplace equation. In this meshless and integration-free method, the harmonic solution representing the steady-state temperature or the electric potential is approximated by a linear combination of non-singular fundamental solutions with sources located outside the closure of the solution domain. The unknown coefficients in this expansion, the points of separation of the Signorini boundary conditions and possibly the unknown Signorini boundary (in the inverse problem) are determined by imposing/collocating the boundary conditions which can be of Dirichlet, Neumann, Cauchy or Signorini type. This results in a constrained minimization problem which is solved using the MATLAB© toolbox routine `fmincon`. Several numerical examples involving both direct and inverse problems are presented and discussed in order to illustrate the accuracy and stability of the numerical method employed.

1. Introduction

Signorini problems model phenomena in which on a known or unknown portion of the boundary, Dirichlet and Neumann conditions alternate in conjunction with certain inequality constraints [4,8]. The problem is further complicated by the fact that the number and the location of the points where the change in the boundary conditions occurs are unknown [28]. Typical applications occur in beach percolation [1], contact problems in electropainting [2] and free surface problems [12].

Since Signorini problems are free boundary value problems, the interest lies primarily on the boundary of the domain of the problem under consideration. For this reason, it is natural to apply boundary methods such as the boundary element method (BEM) for their solution, see, e.g., [3,13,28–30]. In this paper, the problems considered are posed for the Laplace equation in steady-state heat conduction or electrostatics, but a similar formulation occurs for the Lamé system in elasticity.

The method of fundamental solutions (MFS) is a meshfree boundary method which, due to its simplicity and ease of use, is suitable for the solution of problems in complex geometries. In the potential field, the MFS uses the density of the non-singular set of fundamental solutions of the Laplace equation in the set of harmonic functions. As such, it represents the solution as a linear combination of non-singular fundamental solutions with sources located outside the closure of the solution domain. Although the issue of choosing these fictitious locations may represent a drawback, recent studies, [7,20], have proposed variants of the MFS in which the source points are allowed on the boundary of the solution domain itself. The MFS also generates ill-conditioned systems of equations for the unknown coefficients but then, if required, regularization techniques can be employed, [19].

Prior to this study, the MFS has been applied to solve free boundary problems [14,25] and direct Signorini potential problems in both two and three dimensions [23,24], using the FORTRAN NAG routine `E04UPF` [22]. The novelty of this paper is twofold. First, we employ the MATLAB© [21] toolbox routine `fmincon` instead of the NAG routine. Secondly, we investigate for the first time the application of the MFS for solving inverse Signorini potential problems in two dimensions. The MFS has, in recent years, been used extensively for the solution of various types of inverse problems [17,18] but this is, apparently, the first time it is being used for the solution of inverse Signorini problems. In these *inverse geometric problems* the Signorini contact boundary is also unknown and has to be

* Corresponding author.
E-mail addresses: andreask@ucy.ac.cy (A. Karageorghis), amt5ld@maths.leeds.ac.uk (D. Lesnic), marin.liviu@gmail.com, liviu.marin@fmi.unibuc.ro (L. Marin).

http://dx.doi.org/10.1016/j.compstruc.2015.01.002
0045-7949/

determined from Cauchy noisy data measurements of both primary (temperature, potential) and secondary (heat flux, current) variables on the remaining portion of the boundary of the solution domain.

The paper is structured as follows. In Section 2 we describe how the MFS is applied to direct Signorini problems and provide extensive implementational details. The method is then applied to several numerical examples from the literature. In Section 3 we show that the proposed method may be naturally adapted to solve inverse geometric Signorini problems related to non-destructive control processes [5]. The method is then tested on two such problems for retrieving a circular and an elliptic inner boundary. Finally, in Section 4 we provide some conclusions and ideas for future work.

2. Direct Signorini problems

We consider the following problem for the steady-state heat conduction given by the Laplace equation

$$\Delta u = 0 \quad \text{in} \quad \Omega, \tag{1a}$$

for the temperature u subject to the Dirichlet boundary condition

$$u = f_1 \quad \text{on} \quad \Gamma_1, \tag{1b}$$

the Neumann heat flux boundary condition

$$\frac{\partial u}{\partial n} = f_2 \quad \text{on} \quad \Gamma_2, \tag{1c}$$

and the complementarity conditions on the known free boundary Γ_s

$$u = g \quad \text{when} \quad \frac{\partial u}{\partial n} < h, \quad \text{on} \quad \Gamma_s, \tag{1d}$$

or

$$\frac{\partial u}{\partial n} = h \quad \text{when} \quad u < g, \quad \text{on} \quad \Gamma_s, \tag{1e}$$

where $\boldsymbol{n}$ is the outward unit normal to the boundary, $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_s$, $\Gamma_1 \cap \Gamma_2 = \Gamma_1 \cap \Gamma_s = \Gamma_2 \cap \Gamma_s = \emptyset$, and g and h are given functions.

A similar problem can occur in electrostatics, where u denotes the electric potential. For the well-posedness and variational formulation of such a direct Signorini problem see, e.g. [8,10]. Note that it is often convenient to combine (1d) and (1e) in the form, [23],

$$u \leqslant g, \quad \frac{\partial u}{\partial n} \leqslant h, \quad (u - g)\left(\frac{\partial u}{\partial n} - h\right) = 0 \quad \text{on} \quad \Gamma_s. \tag{2}$$

2.1. The method of fundamental solutions (MFS)

In the application of the MFS to the Signorini Problem (1), we seek an approximation to the solution of Laplace's Eq. (1a) as a linear combination of fundamental solutions of the form [9]

$$u_N(\boldsymbol{c}, \xi; \boldsymbol{x}) = \sum_{k=1}^{N} c_k \mathcal{G}(\xi_k, \boldsymbol{x}), \quad \boldsymbol{x} \in \overline{\Omega}, \tag{3}$$

where $\mathcal{G}$ is the fundamental solution of the Laplace equation, which in two dimensions is given by

$$\mathcal{G}(\xi, \boldsymbol{x}) = -\frac{1}{2\pi} \ln|\xi - \boldsymbol{x}|. \tag{4}$$

We choose M_1 collocation points $(\boldsymbol{x}_k)_{k=1}^{M_1}$ on Γ_1, M_2 collocation points $(\boldsymbol{x}_k)_{k=M_1+1}^{M_1+M_2}$ on Γ_2 and M_s collocation points $(\boldsymbol{x}_k)_{k=M_1+M_2+1}^{M_1+M_2+M_s}$ on Γ_s. We denote the total number of collocation points by $M = M_1 + M_2 + M_s$. We also place N singularities $(\xi_k)_{k=1}^{N} \in \mathbb{R}^2 \setminus \overline{\Omega}$ spread uniformly on a pseudo-boundary $\partial\Omega'$ similar to $\partial\Omega$ at a fixed distance d from it in the direction of the outward normal. Clearly, the distance d depends on how far we can harmonically extend the solution u outside $\overline{\Omega}$ and a few guidelines may be found in [27].

2.2. Implementational details

There are N unknowns, namely the coefficients $(c_k)_{k=\overline{1,N}}$ in (3), which can be determined by imposing the boundary conditions (1a–e).

This is achieved by using the MATLAB© routine `fmincon` which finds the constrained minimum of a scalar nonlinear multivariate function. This is in contrast to the previously used routine `lsqnonlin` which minimizes a sum of squares. The choice of `fmincon` was made because of its ability to include both linear and nonlinear constraints thus accommodating the constraints resulting from the complementarity conditions (1d) and (1e).

In particular, the scalar function $F(\boldsymbol{c})$ which is to be minimized by the routine `fmincon` is defined as follows: We take

$$\phi_j(\boldsymbol{c}) = \begin{cases} u_N(\boldsymbol{c}, \xi; \boldsymbol{x}_j) - f_1(\boldsymbol{x}_j), & j = 1, \ldots, M_1, \\ \frac{\partial u_N}{\partial n}(\boldsymbol{c}, \xi; \boldsymbol{x}_j) - f_2(\boldsymbol{x}_j), & j = M_1 + 1, \ldots, M_1 + M_2, \\ (u_N(\boldsymbol{c}, \xi; \boldsymbol{x}_j) - g(\boldsymbol{x}_j))\left(\frac{\partial u_N}{\partial n}(\boldsymbol{c}, \xi; \boldsymbol{x}_j) - h(\boldsymbol{x}_j)\right), & j = M_1 + M_2 + 1, \ldots, M_1 + M_2 + M_s, \end{cases} \tag{5}$$

where $\boldsymbol{c} = (c_1, c_2, \ldots, c_N)$, and $\phi(\boldsymbol{c}) = (\phi_1(\boldsymbol{c}), \phi_2(\boldsymbol{c}), \ldots, \phi_M(\boldsymbol{c}))$, and define

$$F(\boldsymbol{c}) = ||\phi(\boldsymbol{c})||_2^2 = \sum_{j=1}^{M} \phi_j^2(\boldsymbol{c}). \tag{6}$$

In some instances, the minimization of the functional $\widehat{F}(\boldsymbol{c}) = ||\phi(\boldsymbol{c})||$ instead of (6) led to more rapid convergence.

Moreover, we have the $2M_s$ linear constraints

$$u_N(\boldsymbol{c}, \xi; \boldsymbol{x}_j) \leqslant g(\boldsymbol{x}_j) \quad \text{and} \quad \frac{\partial u_N}{\partial n}(\boldsymbol{c}, \xi; \boldsymbol{x}_j) \leqslant h(\boldsymbol{x}_j), \quad j = M_1 + M_2 + 1, \ldots, M_1 + M_2 + M_s, \tag{7}$$

which may be recast in the form

$$A\boldsymbol{c} \leqslant \boldsymbol{b}, \tag{8}$$

where A is a $2M_s \times N$ matrix and $\boldsymbol{b}$ is a $2M_s \times 1$ vector.

More specifically, if we define the two $M \times N$ matrices U and T by

$$U_{i,j} = \mathcal{G}(\xi_j, \boldsymbol{x}_i), \quad T_{i,j} = \frac{\partial \mathcal{G}}{\partial n}(\xi_j, \boldsymbol{x}_i), \quad i = 1, \ldots, M, \; j = 1, \ldots, N, \tag{9}$$

then the matrix A in (8) is given by

$$A_{i,j} = U_{M_1+M_2+i,j}, \quad A_{M_s+i,j} = T_{M_1+M_2+i,j}, \quad i = 1, \ldots, M_s, \; j = 1, \ldots, N. \tag{10}$$

Finally, the vector $\boldsymbol{b}$ in (8) is given by

$$b_i = g(\boldsymbol{x}_i), \quad b_{M_s+i} = h(\boldsymbol{x}_i), \qquad i = 1, \ldots, M_s. \tag{11}$$

In the case of inverse problems investigated in Section 3, the constraints (7) become nonlinear which the routine `fmincon` can accommodate in the form

$$\boldsymbol{C}(\boldsymbol{y}) \leqslant \boldsymbol{0}, \tag{12}$$

where $\boldsymbol{C} = (C_1, C_2, \ldots, C_{2M_s})$ in a $2M_s$-function vector defined by

$$C_j = u_N(\boldsymbol{c}, \xi; \boldsymbol{x}_{M_1+M_2+j}) - g(\boldsymbol{x}_{M_1+M_2+j}),$$

$$C_{M_s+j} = \frac{\partial u_N}{\partial n}(\boldsymbol{c}, \xi; \boldsymbol{x}_{M_1+M_2+j}) - h(\boldsymbol{x}_{M_1+M_2+j}), \quad j = 1, \ldots, M_s,$$

and $\boldsymbol{y}$ is the vector of unknowns.