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A finite element interior-point implementation of tension field theory

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ABSTRACT

This paper presents an interior-point implementation method of the relaxed strain energy function to model the stress distribution in membranes in the presence of wrinkles. The relaxed strain energy function is reformulated using the interior-point method. This formulation is proven to be quasi-convex with respect to the deformation gradient. A set of governing equations for the membrane is developed in the standard finite element format. The solution to this set of equations is obtained using the interior-point method, and several numerical examples are presented for validation. The proposed method is shown to converge and to be robust and efficient.

1. Introduction

Membrane structures have a rich history and are widely used in aerospace and structural engineering applications. A few examples can be found in solar sails, airbags, atmospheric balloons and parachutes. These membranes have many advantages, including their ability to take complex shapes and their low mass to surface ratio, which is especially important in aerospace engineering. Although membranes can carry tensile loads very well, they tend to wrinkle under the slightest compressive load. It is important to be able to predict the stress distribution in the membrane, taking into account the possible presence of wrinkles, as this affects the load carrying capabilities of a membrane.

Wrinkles in a structure are often of a much smaller scale than the structure itself. In order to model these wrinkles the mesh size in e.g. a finite element model needs to be smaller than the wrinkle size, leading to high computational costs. A possible solution is to model the wrinkles as a continuous in-plane contraction of the membrane. The exact geometry of the wrinkles is lost by this procedure, however the stress field in the membrane is modeled accurately, and the computational costs are reduced significantly. This solution, known as the tension field theory, is widely used and was first introduced by Wagner [1].

A few geometrically exact models for membrane wrinkling exist in which the tension field theory method is not applied, yet the information about the detailed wrinkling configuration is maintained. Puntel [2] gives an analytical solution for stretching a flat membrane and an analysis of a point load on a pressurized sphere is given in Vella [3]. Geometrically exact numerical models are presented in Flores et al. [4] and Weinberg et al. [5]. Capturing wrinkling geometry requires the mesh size in a finite element model to be smaller than the wrinkling in the membrane which results in high computational costs for very thin membranes [6]. Additionally the detailed wrinkling geometry of the membrane often is of small importance, therefore the tension field theory method is used in most models for membranes.

The governing equations for the load carrying capabilities of a membrane are presented in the tension field theory method [1]. It is assumed that the membrane has zero bending stiffness and all compressive stresses are eliminated. This elimination is enabled by the introduction of an in-plane contraction which carries zero strain energy. Analytical solutions to these equations exist for a few specific cases. Mansfield [7,8] presents solutions for the tension field of a rectangular strip under shear and an annular ring under torsion loading. These solutions have been validated with experiments and serve well for verification purposes.

Pipkin [9,10] in his work incorporates the tension field theory into the theory of finite deformations as a special case of non-linear elasticity. A modified stored energy function is developed by relaxing the strain energy function of the material to obtain a non-decreasing and convex relaxed strain energy function.

Based on this work Steigmann [11] presents the tension field theory in terms of two potential functions. Different membrane states are derived and it is demonstrated that application of the relaxed strain energy leads to stable solutions for the stress

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distribution. The first numerical results based on the relaxed strain energy function are also presented by Steigmann [12].

Several numerical methods in which the tension field theory method is used have been developed over the years. In one class of methods the projection technique of Akita et al. [13] is implemented. Here the deformation of the membrane is split into a strain carrying and zero-strain carrying part. This is realized by projecting the zero-strain energy components out of the stiffness matrix, which modifies the constitutive law. Jarasjarungkiat et al. [14] have extended this method for use with orthotropic materials.

In a second class of numerical methods for modeling the tension field in membranes the constitutive law is maintained, but the deformation gradient is modified. Roddeman et al. [15] present the theory and analysis for this modification in isotropic materials while Schoop et al. [16] implement it using finite elements. Raible et al. [17] and Epstein [18] extend the method to enable the modeling of orthotropic and general anisotropic materials respectively. Pagitz et al. [6] propose a new finite element that includes a rotational degree of freedom to control the direction of the tension in the membrane. The performance of this element is verified even for the membrane regions close to slacking. Dynamics methods to model tension fields are developed in Kang [19] and Miyazaka [20]. An extension into non-linear material laws to account for plasticity is presented by Mosler et al. [21] using a variational formulation of the internal strain energy. The tension field theory method is formulated using the relaxed strain energy function by Pipkin [10], and an explicit parameterization of the wrinkling strain tensor is applied. The convexity properties of the energy functional are not preserved by this parameterization.

In this paper a new implementation of the relaxed strain energy function by Pipkin [10] will be presented. Convexity properties of the relaxation are preserved and exploited, and the final set of governing equations is presented in the standard finite element format. For this purpose, a reformulation of the relaxed strain energy function will be derived for which the convexity properties yield a computationally efficient method. The resulting reformulation will be a constrained, convex optimization problem, which will be solved using the interior-point method. With this implementation no *if-else* statements are required in the proposed method, as they are in many of available methods to distinguish between the three possible states of a membrane: taut, wrinkled or slack.

The outline of the paper is as follows: the kinematics of membranes is presented in Section 2, which is followed by the general equilibrium conditions in Section 3. These general equilibrium conditions lead to an unstable energy minimization, as shown in Pipkin [10]. Therefore, a variational principle based on the relaxed strain energy function is applied in Section 4 to obtain a stable set of governing equations. The discretization of the membrane structure and the linearization of the equations is performed in Section 5. The numerical implementation of the discretized equations is discussed in Section 6. Several numerical results are presented in Section 7 in which the proposed method is verified and validated. The conclusions are presented in Section 8. In addition, two appendices have been attached for completion. The mathematical conditions for quasi-convexity are presented in Appendix A. These conditions are related to the relaxation of a function in Appendix B, and a closed form expression of this relaxation is derived.

2. Kinematics

Membranes are very thin shells and thus they have a low bending stiffness, which is neglected in the analysis of the tension field in membranes; this causes the membrane to wrinkle under the slightest compressive deformation. In tension field theory these wrinkles are modeled as an in-plane contraction of the membrane, meaning that all strains are membranal. As a result only in-plane strains need to be considered in this section.

Consider the membrane in the reference/undeformed configuration occupies a bounded plane with covariant basis vectors $\boldsymbol{A}_\alpha$ ($\alpha = 1,2$). Note: Greek characters are used to denote the reference configuration. The reference position of any material point can then be given as $\boldsymbol{r}_0 = \boldsymbol{A}_\alpha \xi^\alpha$, with ξ^α being the contravariant components of the material point.

In the current/deformed configuration the membrane can occupy the space normal to the reference configuration which is described by the covariant basis vectors $\boldsymbol{a}_i$ ($i = 1,2,3$). The current position vector then becomes $\boldsymbol{r} = \boldsymbol{a}_i x^i$. Let $G_{\alpha\beta}$ and g_{ij} be the metric tensors in the reference and current configurations respectively:

$$G_{\alpha\beta} = \boldsymbol{A}_\alpha \cdot \boldsymbol{A}_\beta \tag{1}$$

$$g_{ij} = \boldsymbol{a}_i \cdot \boldsymbol{a}_j \tag{2}$$

The deformation from the reference to the current configuration is described by the deformation gradient $\boldsymbol{F} = F^i_\alpha \boldsymbol{A}^\alpha \boldsymbol{a}_i$:

$$F^i_\alpha = \frac{\partial x^i}{\partial \xi^\alpha} \tag{3}$$

This shows that the components of $\boldsymbol{F}$ form a 3×2 matrix, which means $\boldsymbol{F}$ has a null space along the normal of the membrane. The null space is spanned by the vectors $\boldsymbol{u}$ for which the pull back by the deformation gradient equals zero: $\boldsymbol{F}^T\boldsymbol{u} = \boldsymbol{0}$.

The Green–Lagrange strain is a symmetric 2×2 tensor defined in the reference configuration:

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right) \tag{4}$$

In convected coordinates a material point in the current configuration is described by the same set of coordinates as in the reference configuration. The Green–Lagrange strain tensor in convected coordinates is given as:

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{g} - \boldsymbol{G}) \tag{5}$$

Let $\boldsymbol{u}$ be the displacement vector of this material point, then:

$$\boldsymbol{a}_i = \boldsymbol{A}_i + \frac{\partial \boldsymbol{u}}{\partial \xi_i} = \boldsymbol{A}_i + \boldsymbol{u}_{,i}, \quad i = 1,2 \tag{6}$$

The engineering strain in Voigt form then becomes:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \boldsymbol{A}_1 \cdot \boldsymbol{u}_{,1} \\ \boldsymbol{A}_2 \cdot \boldsymbol{u}_{,2} \\ \boldsymbol{A}_1 \cdot \boldsymbol{u}_{,2} + \boldsymbol{A}_2 \cdot \boldsymbol{u}_{,1} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \boldsymbol{u}_{,1} \cdot \boldsymbol{u}_{,1} \\ \boldsymbol{u}_{,2} \cdot \boldsymbol{u}_{,2} \\ \boldsymbol{u}_{,1} \cdot \boldsymbol{u}_{,2} + \boldsymbol{u}_{,2} \cdot \boldsymbol{u}_{,1} \end{pmatrix} \tag{7}$$

3. Equilibrium equations

Let $\phi(\boldsymbol{E}) = \phi(\boldsymbol{E}(\boldsymbol{F})) = \psi(\boldsymbol{F})$ be the strain energy function of the material, which is assumed to be convex in $\boldsymbol{E}$ as is the case when the stress–stain relation is one-to-one [22,10]. The first and second Piola–Kirchhoff stress tensors are given in (8) and (9) respectively:

$$\boldsymbol{P} = P^\alpha_i \boldsymbol{a}_\alpha \boldsymbol{b}^i \quad P^\alpha_i = \frac{\partial \psi}{\partial F^i_\alpha} \tag{8}$$

$$\boldsymbol{S} = S^{\alpha\beta} \boldsymbol{a}_\alpha \boldsymbol{a}_\beta \quad S^{\alpha\beta} = \frac{\partial \phi}{\partial E_{\alpha\beta}} \tag{9}$$

Because of symmetry in $\boldsymbol{E}$, also $\boldsymbol{S}$ is symmetric. The relation between $\boldsymbol{P}$ and $\boldsymbol{S}$ is given as:

$$P^\alpha_i = G_{ij} F^j_\gamma S^{\gamma\alpha} \Rightarrow \boldsymbol{P} = \boldsymbol{FS} \tag{10}$$

The strong form of the equilibrium equations for any material point in the membrane is given as: