Computers and Structures 151 (2015) 115–129

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

Limit analysis of structures: A convex hull formulation

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ARTICLE INFO

Article history:
Received 12 June 2014
Accepted 13 January 2015

Keywords:
Convex hull
Linear programming
Limit analysis
Combined stresses

ABSTRACT

Limit analysis is treated herein in the framework of mathematical programming introducing a convex hull formulation for expressing the yield conditions in static and kinematic theorem. The proposed formulation differs in the number of variables and yield constraints compared to the standard one, which expresses yield condition as the intersection of halfspaces. The two formulations are compared in terms of computational efficiency. Numerical results of plane steel frames prove the computational advantages of convex hull formulation for both 2D and 3D stress resultant interaction and demonstrate the effect of combined stresses on the load carrying capacity.

1. Introduction

Plastic analysis can be performed either by a step-by-step method following the succession of the inelastic structural response at critical sections or directly via limit analysis techniques. Step-by-step method is physically more informative providing the entire evolution until structural collapse, while the ultimate state (which is of primal interest from engineering perspective) can be obtained almost instantaneously via limit analysis methods. For linearized yield criteria and rigid-perfectly plastic behavior, as well as hardening behavior with unbounded plastic deformations, limit analysis can be cast as a Linear Programming (LP) problem that still remains computationally advantageous. The use of LP and its duality offer the supportive mathematical structure for the two theorems of limit analysis, i.e. the static (lower bound) theorem and the kinematic (upper bound) theorem. The first approaches the true load factor from below for statically admissible trials that satisfy equilibrium and yield conditions, while the second determines an upper bound of the load factor among kinematically admissible solutions that are stressed within the yield limits [1,2].

Incorporation of Linear Programming into limit analysis was introduced by Charnes and GreenBerg [3] for the ultimate state analysis of trusses. A variety of alternative mathematical programming procedures for limit analysis of discrete structures described by piecewise linear (PWL) elastic-perfectly plastic constitutive laws were formulated and compared with respect to their computational merit by Maier et al. [4]. An alternative yield approach using the vertices of yield polyhedron was proposed by Zavelani et al. in 1974. This approach, denoted as “vertex” or “corners” formulation, was used for shakedown finite element analysis [5] and plane limit analysis [6,7]. The effect of combined stresses on the ultimate state of structures was further addressed by Polizzotto [8] and generalized by Grierson and Aly [9]. Methods that improve limit load estimation of rigid-perfectly plastic structures under the effect of combined stresses were proposed by Tin-Loi [10,11] and Ardito et al. [12]. The formulation was extended to address holonomic and nonholomic behavior accounting also for hardening/softening by Maier et al. [13–16] and Tangaramvong and Tin-Loi [17,18]. Most of this development is based on a piecewise linearization of the convex yield surface that delimits the elastic domain as an intersection of half-spaces determined by a number of hyperplanes. More recently, methods for approximating the yield surface with ellipsoids were proposed forming second-order cone programming (SOCP) problems by Skordeli and Bisbos [19] and Bleyer and Buhan [20].

In this work, yield polyhedron is expressed in the mathematical context of convex hull and is compared to hyperplane (standard) formulation. According to the latter, yield constraints are introduced into the problem through the equations of the linear segments/hyperplanes that are used for the approximation of the nonlinear yield surface. Convex hull formulation, on the other hand, expresses the yield condition in the form of a linear combination of all vertices that define the linearized yield surface. The two expressions of yielding lead to the corresponding primal and dual problems that differ in number of variables and yield constraints. The primal formulations are used and compared in terms of computational efficiency for limit analysis of plane frames, while the effect of combined stresses is also examined.

The organization of the paper is as follows. First, the governing relations of the static and kinematic theorem of limit analysis,

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http://dx.doi.org/10.1016/j.compstruc.2015.01.014
0045-7949/

namely equilibrium, yield conditions and compatibility relations, are presented for plane frames. Yielding is established in two alternative ways, i.e. using the equations of linear segments/planes and a convex hull formulation. Then, the static and kinematic theorems of limit analysis are formulated as primal- dual LP problems using both expressions for yield conditions. Moreover, two interaction cases for stress resultants are considered, namely axial force-bending moment (NM) and axial-shear force-bending moment (NQM) interaction. Finally, numerical examples for 2D steel frames are presented that illustrate the effect of combined stresses on the maximum load factor and the yielding pattern. These are further used to compare the two formulations of the yield surface in terms of their computational efficiency.

2. Basic assumptions

Plane frames are considered herein consisting of prismatic elements subjected only to nodal loading for simplicity reasons. Moreover, small displacements are assumed to establish equilibrium equations at the initial undeformed configuration. In addition, plastic behavior, if present, is considered only at preselected critical sections, i.e. the end sections of the elements, whereas the remaining parts behave elastically. Yield conditions are appropriately linearized and the behavior of all critical sections is considered rigid-perfectly plastic.

Matrix notation is adopted throughout. Matrices are represented by capital bold-face letters, while vectors by lowercase bold characters.

3. Equilibrium

Each plane beam element develops six stress resultants at its ends, as shown in Fig. 1. Herein, the axial force (s_1^i), bending moment at the start node j (s_2^i) and bending moment at the end node k (s_3^i), are considered as independent primary actions for member i [18]. Thus the six end actions of the element can be expressed at the global axes system in terms of the local basic actions by using the corresponding equilibrium matrix as follows:

$$\begin{Bmatrix} F_x^j \\ F_y^j \\ M^j \\ F_x^k \\ F_y^k \\ M^k \end{Bmatrix} = \begin{bmatrix} \cos\omega^i & -\sin\omega^i/L^i & -\sin\omega^i/L^i \\ \sin\omega^i & \cos\omega^i/L^i & \cos\omega^i/L^i \\ 0 & 1 & 0 \\ -\cos\omega^i & \sin\omega^i/L^i & \sin\omega^i/L^i \\ -\sin\omega^i & -\cos\omega^i/L^i & -\cos\omega^i/L^i \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} s_1^i \\ s_2^i \\ s_3^i \end{Bmatrix} = \boldsymbol{B}^i \cdot \boldsymbol{s}^i \tag{1}$$

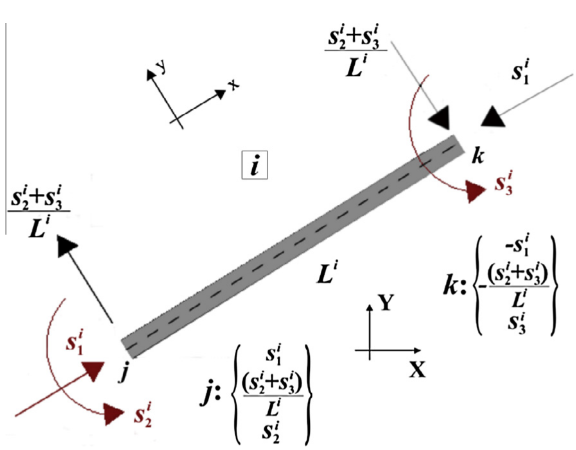

Fig. 1. Frame element i with equilibrated stress resultants-end actions.

where F_x^j, F_y^j, M^j are the global X and global Y forces and bending moment at the start node and F_x^k, F_y^k, M^k are the actions at the end node of the element i at the global system, ω^i is the angle formed rotating the global X-axis counterclockwise to meet the local x-axis and L^i is the element length, $\boldsymbol{B}^i$ is the (6×3) equilibrium matrix of the element and $\boldsymbol{s}^i$ is the (3×1) stress vector of the element.

The equilibrium for the whole structure is then established in terms of the unknown vector of stresses of all members as:

$$\boldsymbol{B} \cdot \boldsymbol{s} = a \cdot \boldsymbol{f} + \boldsymbol{f}_d \tag{2}$$

where $\boldsymbol{B}$ is the $(n_f \times 3n_{el})$ structural equilibrium matrix, assembled by the corresponding element equilibrium matrices arranged in a block diagonal manner, $\boldsymbol{s}$ is a $(3n_{el} \times 1)$ vector of all stresses in local systems, a is a scalar load factor, $\boldsymbol{f}$ the $(n_f \times 1)$ vector of nodal loading in the global system, $\boldsymbol{f}_d$ is the $(n_f \times 1)$ fixed nodal load vector, n_{el} denotes the number of elements and n_f the number of degrees of freedom.

4. Yield condition for multi-component interaction

4.1. Hyperplane equations – standard formulation

The nonlinear yield criterion is a priori linearized forming a polyhedron to facilitate expressing the yield condition as a set of linear constraints. The elastic domain is denoted by the common space of all halfspaces in the form [21]:

$$\{\boldsymbol{s}_d | \mathbf{a}^T \cdot \boldsymbol{s}_d \leqslant \boldsymbol{r}_d\} \tag{3}$$

where $\mathbf{a}$ is the unit normal vector of the hyperplane, $\boldsymbol{s}_d$ is the vector of normalized stresses and $\boldsymbol{r}_d$ determines the offset of the hyperplane from the origin.

Yield condition is defined in this section as a set of a finite number of linear inequalities, which geometrically represent the intersection of a finite number of halfspaces and hyperplanes.

In general, considering the interaction of d number of stress resultants (d-component interaction) and the yield surface of dimension d is approximated with h hyperplanes, the yield condition for all critical sections of the whole frame is formed in terms of stresses $\boldsymbol{s}$ as:

$$\boldsymbol{N}^T \cdot \boldsymbol{s} \leqslant \boldsymbol{r} \tag{4}$$

where $\boldsymbol{N}$ is the $(3n_{el} \times 2hn_{el})$ matrix of all scaled -with respect to yield capacities of stresses-normal vectors and $\boldsymbol{r}$ is the $(2hn_{el} \times 1)$ vector that includes the yield limits of all yield lines [2]. Relation (4) is analyzed in detail for 2D (axial force-bending moment) and 3D (axial-shear force-bending moment) interaction in Sections 7.2 And 7.3 respectively.

4.2. Convex hull formulation

4.2.1. Mathematical considerations

The convex hull of a set of points or vertices is the domain within and on the envelope formed by the outer vertices. Mathematically a set C is convex if the line segment between any two points in C lies in C, i.e., if for any x_1, $x_2 \in C$ and any θ with $0 \leqslant \theta \leqslant 1$, $\theta \cdot x_1 + (1-\theta) \cdot x_2 \in C$. Furthermore, a point of the form $\theta_1 \cdot x_1 + \ldots + \theta_n \cdot x_n$, where $\theta_1 + \ldots + \theta_n = 1$ and $\theta_i \geqslant 0$, $i = 1 \ldots n$, is a convex combination of the points-vertices $x_1, \ldots, x_n$ [21].

The convex hull of a set of points C (Fig 2a), denoted by $\textbf{\textit{conv}}\ C$, is the set of all convex combinations of points in C:

$$\textbf{\textit{conv}}\ C = \{\theta_1 x_1 + \cdots + \theta_n x_n | x_i \in C, \quad \theta_i \geqslant 0, i = 1, \ldots, n, \\ \theta_1 + \cdots + \theta_n = 1\} \tag{5}$$

where θ_i are nonnegative coefficients and $x_1, \ldots, x_n$ are the points-vertices. The convex hull or convex envelope of set C is the smallest convex set that contains C [21].