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Definition of warping modes within the context of a higher order thin-walled beam model

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ABSTRACT

A procedure for the definition of uncoupled warping modes within the framework of a higher order beam model is presented. An approximation of the displacement field over the cross-section by a set of linearly independent basis functions is considered in the models formulation so as to capture 3D structural phenomena. By considering the cross-section in-plane rigid, a linear eigenvalue problem stemming from the models governing equations is derived, allowing to retrieve classic solutions and to derive a set of hierarchical warping modes. Numerical examples are presented in order to verify the ability of the model to simulate the warping of thin-walled structures.

1. Introduction

The structural analysis of thin-walled prismatic structures through one-dimensional models requires the consideration of higher order deformation modes in order to accurately represent its three dimensional structural behaviour. In fact, being those models derived by reducing the three-dimensional elasticity equations to a set of equations defined along the member axis by an appropriate projection of the displacement field, the definition of a convenient set of basis functions is essential to capture the 3D structural behaviour.

The warping of thin-walled structures with open cross-sections was considered by [1] through the definition of an additional coordinate for the approximation of the displacement field, the so-called sectorial coordinate. The cross-section was assumed to be in-plane undeformable and the shear deformation of the middle surface was neglected. A theory for the non-uniform torsion of closed cross-section was derived in [2,3], considering the shear strain of the cross-section midline to be given by the *Saint-Venant* uniform torsion theory. A more accurate theory for the non-uniform torsion of closed thin-walled beams was derived in [4,5] by considering an additional term representing the cross-section warping, which has an amplitude variable along the beam axis that is not related with the cross-section angle of torsion; reference should be made that this approach was also considered in [1] for solid sections. This theory is established and known as Benscoter theory, [6]. The cross-section "classic" warping is made orthogonal to the flexural modes by considering an adequate position for measuring the sectorial coordinate that defines the warping function, which is similar to consider the beam flexure around the principal axes in order to uncouple the axial force and bending moments.

A general theory for the analysis of thin-walled beams applied to a cross-section either with an open, closed or branched midline profile was presented in [7–9]. The theory relies on the assumption of the *Vlassov* hypothesis, considering only the shear strain corresponding to the theory of torsion by *Saint-Venant* for closed cross-section, [3]. The displacement field is approximated in terms of the derivative of its tangential components along the beam axis, being the axial displacements obtained through the hypothesis of neglecting the membrane shear strain, corresponding to a procedure already adopted by [10]. However, an orthogonality criterion similar to the classic warping was not possible to derive; instead, an uncoupling procedure based on an attempt do diagonalize the beam governing equations through generalised eigenvalue problems was adopted. A development of thin-walled beam models to the analysis of bridge structures considering warping modes was presented in [11–13], considering some of the formulations reported in [14] that follow the seminal work of [7–9,15].

The generalised beam theory (GBT) developed by [16,17] allows the definition of the cross-section deformation field through a cross-section discretization in terms of axial displacements, obtaining the transverse displacement by neglecting the shear strain. GBT was extended so as to consider (i) the shear-lag effect by adopting a set of shear-lag warping modes, [15,18], and (ii) the application to cross-sections with a generic midline profile geometry; recall that the initial formulation of GBT was developed for open cross-sections. Recently, further developments of the theory were made,

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allowing its application to cross-sections with an open, closed or branched geometry, [19–22]. Regarding the uncoupling between deformation modes, the theory considers the diagonalization of the beam governing equations as a criteria, adopting towards this end a set of justified generalised eigenvalues problems.

Several other beam formulations accounting for the warping of thin-walled structures have been developed. Although, some of them were developed towards the application to specific structural behaviours, it is worth to mention some of the respective concepts.

A thin-walled beam formulation for anisotropic materials considering the out of plane warping of beam cross-sections but assuming its in-plane rigidity was presented by [23]. In that formulation, a set of orthogonal functions for the shear dependent warping of thin-walled beams was derived, being defined the corresponding weak form of the equilibrium equations. A set of orthogonal warping functions, defining the so-called *eigenwarpings*, were obtained from the solution of a generalised eigenvalue problem associated with the equilibrium differential equations. An “improved Bernoulli model” and an “improved Saint-Venant model” have been defined by adding to the Bernoulli model (linear axial displacement) and to the Saint-Venant model (linear axial stress distribution) a series expansion of these eigenwarpings.

The warping of thin-walled beams was also taken in account in [24] by considering an approximation of the thin-walled axial displacement field through a linear combination of basis functions only dependent of the cross-section coordinate. The basis functions are considered to be orthogonal to the uniform and linear functions associated with the translation and rotation of the cross-section. Comparatively to the model of [23], the formulation of [24] derives the system of governing equations from the assumptions made on the complete description of the displacement field, whereas in [23] the solution of warping is obtained separately for a specific displacement interpolation.

A thin-walled beam model applicable to cross-sections with both open and closed midline profiles, considering the cross-section in-plane rigid but including the membrane shear deformation, was presented in [25]. A shear flexible element considering warping was presented by [26], being the model applicable to thin-walled beams with an open profile of an arbitrary geometry. In terms of kinematics, the formulation considers the axial displacements of the cross-section to be defined through the linear combination of the axial displacement of the centroid, the cross-section rotations due to the flexure and the warping function defined through the sectorial coordinate. A finite element is derived considering the approximation of the displacement, rotation and warping through linear, quadratic and cubic functions, respectively.

A formulation for the analysis of thin-walled rectangular hollowed cross-sections subjected to torsion is developed in [27] in the sequel of a previous work, [28]. A set of orthogonal basis functions is adopted for the axial displacement of the cross-section webs, being the displacement of the flanges obtained from compatibility requirements. An equilibrium equation governing the warping of the walls is established in terms of the basis functions coordinates, which is proven to be independent of the angle of twist. On the other hand, an equation similar to the non-uniform torsion theory is derived, being the angle of twist coupled with the warping functions parameters.

A procedure for the definition of warping functions was presented in[29–32] within the framework of a beam model formulation. The beam model considers a division of the cross-section into rectilinear elements, being adopted a linear variation of the displacements along each wall element. The axial displacements are obtained adding to the Bernoulli displacements a linear combination of these additional warpings.

A thin-walled beam model that assumes the cross-section in-plane rigid but considers the out-of-plane warping and the “membrane” shear deformation is presented in this paper. The classic equations are retrieved side by side with a set of governing equations representing higher order deformations. In a previous work by the authors, [33], the warping modes were obtained from the solution of the quadratic eigenvalue problem associated with the differential equations of the beam model. In the sequel of a work presented by the authors in [34], an efficient, simple and innovative procedure for the definition of higher order warping modes is herein presented, being verified that these modes allow to accurately represent the higher order warping of in-plane deformable cross-sections. Towards the definition of orthogonal warping modes, the higher order beam model differential equations are rewritten in terms of axial displacements by considering the cross-section in-plane rigid, allowing to obtain a set of uncoupled warping modes directly from the corresponding linear symmetric eigenvalue problem. These modes are uncoupled according to bi-orthogonality conditions associated with linear symmetric eigenvalue problems, which allow to separate the structural phenomena of thin-walled structures. A comparison with the results of a higher order beam theory proposed by [35], which identifies similar warping modes but for an in-plane deformable cross-section, was successfully performed.

2. Model formulation

A one dimensional model for the analysis of thin-walled beams in order to consider the corresponding out-of-plane warping is developed. The formulation considers the cross-section to be in-plane rigid and includes the corresponding shear deformation. Towards an efficient approximation of the displacement field, the cross-section is divided into laminar elements, being the displacement field approximated for each element along the corresponding middle surface (see Fig. 1).

2.1. Displacement field

The displacement field is defined admitting the beam cross-section to be “divided” into n laminar elements without any geometrical restriction, i.e., it is possible to deal with more than two walls converging in a node as well as to consider two consecutive aligned walls. The displacement components are defined through a set of interpolation functions independently of the corresponding direction. A local reference frame $O(x,s,n)$ is adopted, being the beam longitudinal axis represented by x, whereas n represents the perpendicular direction relatively to the wall and s the running coordinate along the cross-section midline profile. The middle surface is therefore defined by the cartesian pair (x,s). A cross-section discretization is represented in Fig. 2.

The structural behaviour of the thin-walled beam is reduced to the cross-section midline profile considering only the corresponding membrane behaviour inasmuch as the cross-section in-plane deformation is disregarded and hence the plate behaviour of the wall is neglected. The displacement field is defined as follows:

$$u_x(x,s,n) = \tilde{u}_x(x,s) = \mathbf{\Phi}\mathbf{u}_x \quad \text{and} \quad u_s(x,s,n) = \tilde{u}_s(x,s) = \mathbf{\Psi}\mathbf{u}_s \tag{1}$$

where $\mathbf{\Phi}$ and $\mathbf{\Psi}$ represent the arrays grouping the corresponding interpolation functions, being $\tilde{u}_x$ and $\tilde{u}_s$ the respective amplitudes. A set of p and m linear independent approximation functions is considered:

$$\mathbf{\Phi} = [\phi_1,\ldots,\phi_p] \quad \text{and} \quad \mathbf{\Psi} = [\psi_1,\ldots,\psi_m] \tag{2}$$

being the respective amplitudes given by:

$$\mathbf{u}_x = [u_{x1},\ldots,u_{xp}]^t \quad \text{and} \quad \mathbf{u}_s = [u_{s1},\ldots,u_{sm}]^t \tag{3}$$