Computers and Structures 147 (2015) 250–258

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

Refined 2-D theories for free vibration analysis of annular plates: Unified Ritz formulation and numerical assessment ☆

Lorenzo Dozio

Department of Aerospace Science and Technology, Politecnico di Milano, via La Masa, 34, 20156 Milano, Italy

ARTICLE INFO

Article history:
Accepted 2 October 2014
Available online 6 November 2014

Keywords:
Free vibration analysis
Circular and annular plates
Higher-order plate theories
Variable-kinematic Ritz method

ABSTRACT

This paper presents a unified Ritz-based method for the computation of modal properties of both thick and thin, circular and annular isotropic plates with different boundary conditions. The solution is based on an appropriate and simple formulation capable of handling in an unified way a large variety of two-dimensional higher-order plate theories. The formulation is also invariant with respect to the set of Ritz admissible functions. In this work, accurate upper-bound vibration solutions are presented by using kinematic models up to sixth order and products of Chebyshev polynomials and boundary-compliant functions. Considering the circumferential symmetry of annular plates and the 2-D nature of underlying theories, the present method is also computationally efficient since only single series of trial functions in the radial direction are required.

1. Introduction

Circular and annular plates are widely adopted as structural elements in many engineering fields. Therefore, reliable mathematical models capable of predicting with high accuracy their dynamic behavior can be of great importance in the design process.

It is known that the accuracy in the computation of natural frequencies and mode shapes of plate structures can strongly depend on the kinematics assumed to represent their deformation. Modeling approaches range from fully three-dimensional (3-D) models, without any simplifying assumption on the kinematics of deformation, to traditional plate theories, like classical plate theory (CPT) and first-order shear deformation theory (FSDT), based on a reduction of the 3-D problem to simple and economical two-dimensional (2-D) models [1]. Many attempts lying in the middle have also appeared in the last three decades. They fall into the category of so-called refined or higher-order plate theories, where the conventional kinematics of FSDT is enriched with various higher-order terms as power series expansion of the thickness coordinate [2–8]. The aim of such refined theories is twofold. Firstly, to preserve the 2-D nature of the model and thus avoid the complexity and computational inefficiency of 3-D elasticity solutions. Secondly, to improve, compared to classical theories, the capability of estimating the correct mechanical behavior of the plate when thickness-to-length ratio increases, accurate through-the-thickness distribution of displacements and stresses is sought or discrete medium-to-high frequency analysis is required.

In contrast to CPT and FSDT, plate theories of high order typically involve complicated mathematical formulations. Derivation and computer implementation of the corresponding models would be less cumbersome with the availability of appropriate techniques capable of handling in an easy and efficient way arbitrary refinements of classical theories. Furthermore, it would be highly desirable to rely on an unified modeling framework giving the ability of performing comparisons of different theories of increasing complexity without the need of a new modeling effort each time.

In view of the above remarks, this paper presents a unified Ritz-based formulation based on an entire class of 2-D higher-order theories for free vibration analysis of both thick and thin isotropic annular plates with different combinations of classical boundary conditions. The novelty of the present work is twofold.

Firstly, a comprehensive assessment of refined plate theories against free vibration of annular plates of any thickness is presented for the first time. Indeed, most of the past investigations on free vibration of circular and annular plates performed an exact or numerical analysis on the basis of CPT and FSDT (see, e.g., [9–13]). A satisfactory number of papers that carried out a 3-D vibration analysis are also available [14–18]. Conversely, probably due to the mathematical and computational complexities mentioned above, higher-order plate theories were employed only in very few works [19–21]. In particular, remarkable exact closed-form frequency solutions are obtained in [20,21] using Reddy's third-order shear deformation theory (TSDT). However, since TSDT

☆ Paper reference number S2012/2012/00020.
E-mail address: lorenzo.dozio@polimi.it

http://dx.doi.org/10.1016/j.compstruc.2014.10.015
0045-7949/

discards thickness-stretching effects, which are increasingly important as the thickness-to-radius increases, their analysis is limited to moderately thick plates. The current study aims at evaluating how accurate natural frequencies of higher-order 2-D theories would be in representing a 3-D problem.

Secondly, all previous works on free vibration of circular and annular plates modeled according to 2-D theories suffer from a common shortcoming: they rely on axiomatic models with a fixed kinematic theory. As a result, the development of a refined theory of a certain order requires each time a new mathematical effort along with the related code implementation. This process can be cumbersome and prone to errors. The powerful yet simple method presented in the following overcomes the above deficiency.

The present study can be considered as an extension to annular plates of the variable-kinematic Ritz method developed in [22–24], which were focused on straight-sided quadrilateral plates. The formulation has some attractive properties. It is invariant with respect to both the specific plate theory and the set of admissible functions. In other words, a unified modeling framework is derived in terms of simple modeling kernels, called *Ritz fundamental nuclei*, which are properly expanded to yield the mass and stiffness matrices of the model. Considering the circumferential symmetry of circular plates and the 2-D nature of the underlying theories, the present method is also computationally efficient since only single series of trial functions in the radial direction are required. In addition, relying on a global approximation, the method has a high spectral accuracy and converges faster than local methods such as finite elements. As a result, the formulation derived in this work is accurate in providing benchmark values yet efficient to be used for design purposes and parametric analysis.

The current paper is an extended version of the conference paper [25] and includes a more complete numerical analysis with new comparison studies for plates with different thickness-to-radius ratios and boundary conditions. The paper is organized as follows. Section 2 contains the mathematical derivation of the method. Details about the Ritz trial set adopted in this study are also given. The convergence and numerical stability properties of the current approach are presented in Section 3. Upper-bound vibration solutions based on various higher-order 2-D models are shown in Section 4. In-depth discussion is provided by comparison the frequency parameters obtained by the current method with various results available in the literature. Finally, some concluding remarks are drawn in Section 5.

2. Theoretical formulation

An annular isotropic plate of outer radius R_o and inner radius R_i is considered as shown in Fig. 1. The plate has uniform thickness h. An orthogonal cylindrical coordinate system is defined with radial direction r ($R_i \leqslant r \leqslant R_o$), circumferential direction θ ($0 \leqslant \theta \leqslant 2\pi$) and thickness direction z ($-h/2 \leqslant z \leqslant h/2$).

For generality and convenience, the present formulation is derived using a dimensionless coordinate ξ ($-1 \leqslant \xi \leqslant 1$) for the radial direction defined as follows

$$\xi = \frac{r}{\gamma} - \delta \tag{1}$$

where

$$\gamma = \frac{R_o - R_i}{2} \tag{2}$$

$$\delta = \frac{R_o + R_i}{R_o - R_i} \tag{3}$$

The displacement vector $\mathbf{u} = \mathbf{u}(\xi, \theta, z, t)$ of a generic point of the plate is given by

$$\mathbf{u}(\xi, \theta, z, t) = \begin{Bmatrix} u_\xi(\xi, \theta, z, t) \\ u_\theta(\xi, \theta, z, t) \\ u_z(\xi, \theta, z, t) \end{Bmatrix} \tag{4}$$

Strain components can be grouped into an in-plane strain vector $\boldsymbol{\varepsilon}_\mathrm{p}$ and out-of-plane (normal) strain vector $\boldsymbol{\varepsilon}_\mathrm{n}$ as follows

$$\boldsymbol{\varepsilon}_\mathrm{p} = \begin{Bmatrix} \varepsilon_{\xi\xi} \\ \varepsilon_{\theta\theta} \\ \gamma_{\xi\theta} \end{Bmatrix} \qquad \boldsymbol{\varepsilon}_\mathrm{n} = \begin{Bmatrix} \gamma_{\xi z} \\ \gamma_{\theta z} \\ \varepsilon_{zz} \end{Bmatrix} \tag{5}$$

Within the framework of linear, small strain, elasticity theory, strain vectors are related to displacements through the following equations

$$\boldsymbol{\varepsilon}_\mathrm{p} = \boldsymbol{D}_\mathrm{p}\mathbf{u} \tag{6}$$

$$\boldsymbol{\varepsilon}_\mathrm{n} = \boldsymbol{D}_\mathrm{n}\mathbf{u} + \boldsymbol{D}_z\mathbf{u} \tag{7}$$

where

$$\boldsymbol{D}_\mathrm{p} = \begin{bmatrix} \left(\frac{1}{\gamma}\right)\frac{\partial}{\partial\xi} & 0 & 0 \\ \left(\frac{1}{\gamma}\right)\frac{1}{\xi+\delta} & \left(\frac{1}{\gamma}\right)\frac{1}{\xi+\delta}\frac{\partial}{\partial\theta} & 0 \\ \left(\frac{1}{\gamma}\right)\frac{1}{\xi+\delta}\frac{\partial}{\partial\theta} & \left(\frac{1}{\gamma}\right)\left[\frac{\partial}{\partial\xi} - \frac{1}{\xi+\delta}\right] & 0 \end{bmatrix} \tag{8}$$

$$\boldsymbol{D}_\mathrm{n} = \begin{bmatrix} 0 & 0 & \left(\frac{1}{\gamma}\right)\frac{\partial}{\partial\xi} \\ 0 & 0 & \left(\frac{1}{\gamma}\right)\frac{1}{\xi+\delta}\frac{\partial}{\partial\theta} \\ 0 & 0 & 0 \end{bmatrix} \tag{9}$$

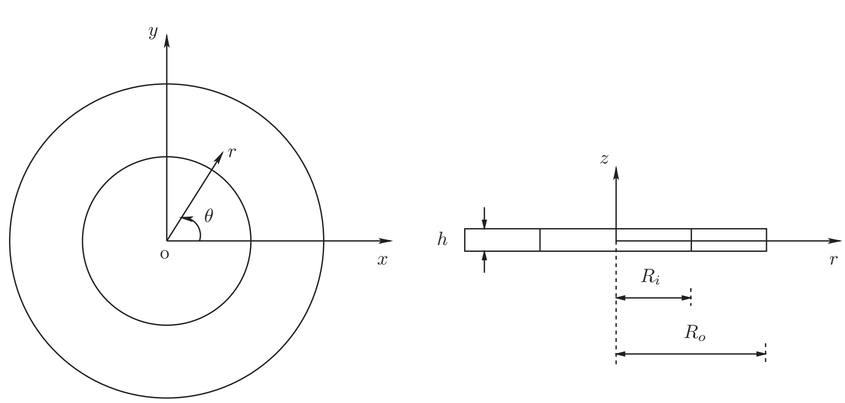

Fig. 1. Geometry of an annular plate.