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On the consistent formulation of torques in a rotationless framework for multibody dynamics

P. Betsch *, N. Sänger

University of Siegen, Paul-Bonatz-Straße 9-11, 57068 Siegen, Germany

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ABSTRACT

A rotationless formulation of flexible multibody dynamics in terms of natural coordinates is considered. Since natural coordinates do not comprise rotational parameters, the consistent formulation and numerical discretization of actuating torques becomes an issue. In particular, the straightforward time discretization of the forces conjugate to natural coordinates may lead to a significant violation of the balance law for angular momentum. The present work shows that the theory of Cosserat points paves the way for the consistent incorporation and discretization of actuating torques. The newly proposed method adds to the energy–momentum consistent numerical integration of flexible multibody dynamics.

1. Introduction

The present work deals with a rotationless description of flexible multibody dynamics that circumvents the use of rotational parameters (Betsch et al. [1,2]). The present approach relies on the canonical embedding of the rotation group into a nine-dimensional linear space. Accordingly, the orientation of a rigid body in space is characterized by nine direction cosines which define a director triad fixed at the rigid body and moving with it (Betsch and Steinmann [3]). The nine direction cosines play the role of redundant coordinates subject to six independent constraints enforcing the orthonormality of the director triad.

A similar approach can be applied to Cosserat solids such as shear deformable beams and shells. In Betsch and Steinmann [4,5] and Betsch and Sänger [6] the rotationless formulation and numerical discretization of geometrically exact Cosserat beams and shells is treated. The advantages of Cosserat solids for the description of flexible multibody systems are emphasized as well in the works by Géradin and Cardona [7], Ibrahimbegović and Mamouri [8], and Bauchau [9].

In this connection, structure-preserving time-stepping methods such as energy–momentum schemes are considered important due to their enhanced numerical stability and robustness, see Géradin and Cardona [7, Chapter 12], Ibrahimbegović et al. [10], Bathe [11], and Bauchau [9, Chapter 17]. It is worth noting that the rotationless formulation of flexible multibody dynamics makes possible the straightforward design of structure-preserving time-stepping schemes such as energy–momentum schemes and momentum-symplectic integrators (Leyendecker et al. [12] and Betsch et al. [13]).

On the other hand the nonstandard rotationless description of rigid bodies and Cosserat solids requires some care concerning the consistent application of actuating torques. The present rigid body formulation falls into the framework of natural coordinates which have a long tradition in multibody system dynamics (see García de Jalón [14] and the references cited therein). By definition, natural coordinates are comprised of Cartesian components of unit vectors and Cartesian coordinates. It is worth noting that our specific choice of natural coordinates (Betsch and Steinmann [3]) has its roots in theoretical mechanics (Saletan and Cromer [15, Chapter 5]).

Using natural coordinates, the application of external torques becomes an issue since conjugate rotational parameters are not available. One way to resolve this issue is the introduction of additional coordinates which are appended to the natural coordinates via specific algebraic constraints (García de Jalón [14] and Uhlar and Betsch [16]).

Alternatively, the redundant forces conjugate to the natural coordinates can be used to take into account the action of external torques. In the present work we focus on this approach. We show that the straightforward time discretization of the forces conjugate to natural coordinates may lead to a significant violation of the balance law for angular momentum. To remedy the situation we recast the rotationless formulation of rigid bodies in terms of skew

* Corresponding author. Address: Department of Mechanical Engineering, University of Siegen, Paul-Bonatz-Straße 9-11, 57068 Siegen, Germany.

E-mail addresses: peter.betsch@uni-siegen.de (P. Betsch), saenger@imr.mb.uni-siegen.de (N. Sänger).

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coordinates. This approach paves the way for the consistent time discretization of the equations of motion. It is worth noting that our newly proposed method has been guided by the close connection between natural coordinates and the theory of Cosserat points (Rubin [17]).

An outline of the rest of the paper is as follows. In Section 2 the equations of motion providing the framework for the present description of flexible multibody systems are summarized. The formulation of rigid body dynamics in terms of natural coordinates is dealt with in Section 3. The extension of the present approach to multibody dynamics is illustrated in Section 4 with the formulation of lower kinematic pairs. After a summary of the main features of the present approach in Section 5, the structure-preserving discretization in time is dealt with in Section 6. To demonstrate the capability of the proposed method two numerical examples are presented in Section 7. Eventually, conclusions are drawn in Section 8.

2. Equations of motion

We start with the equations of motion pertaining to a finite-dimensional mechanical system subject to holonomic constraints. From the outset we confine ourselves to mechanical systems whose kinetic energy can be written as

$$T(\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}} \cdot \boldsymbol{M}\dot{\boldsymbol{q}} \tag{1}$$

Here, $\boldsymbol{q} \in \mathbb{R}^n$ is the vector of redundant coordinates and a superposed dot denotes the derivative with respect to time. Moreover $\boldsymbol{M} \in \mathbb{R}^{n\times n}$ is a **constant** mass matrix. As has been outlined in the Introduction a constant mass matrix is a consequence of the use of natural coordinates for the description of spatial multibody systems. The equations of motion pertaining to the discrete mechanical systems of interest can be written in variational form

$$G^{\delta} = \delta\boldsymbol{q} \cdot \left(\boldsymbol{M}\ddot{\boldsymbol{q}} + \sum_{l=1}^{m} \lambda^l \nabla g_l(\boldsymbol{q}) - \boldsymbol{F}\right) = 0 \tag{2}$$

which has to be satisfied for arbitrary $\delta\boldsymbol{q} \in \mathbb{R}^n$. The last equation has to be supplemented with algebraic constraint equations $g_l(\boldsymbol{q}) = 0$, $1 \leqslant l \leqslant m$. The associated constraint forces assume the form $\sum \lambda^l \nabla g_l(\boldsymbol{q})$, where λ^l are Lagrange multipliers. The last term in (2) accounts for external forcing. For simplicity of exposition we do not distinguish between forces that can be derived from potentials and nonpotential forces. Note, however, that we may replace $\boldsymbol{F} \in \mathbb{R}^n$ in (2) with

$$\boldsymbol{F} \to \boldsymbol{F} - \nabla U(\boldsymbol{q}) \tag{3}$$

Then, the potential forces are derived from a potential function $U(\boldsymbol{q})$, and the nonpotential forces are contained in $\boldsymbol{F}$. Due to the presence of algebraic constraints the equations of motion assume the form of differential–algebraic equations (*DAEs*). The configuration space of the constrained mechanical systems under consideration is defined by

$$\mathrm{Q} = \{\boldsymbol{q} \in \mathbb{R}^n | g_l(\boldsymbol{q}) = 0,\ 1 \leqslant l \leqslant m\} \tag{4}$$

Throughout this work we assume that the constraints are independent. Consequently, the vectors $\nabla g_l(\boldsymbol{q}) \in \mathbb{R}^n$ are linearly independent for $\boldsymbol{q} \in \mathrm{Q}$. Due to the presence of m geometric constraints the discrete mechanical system under consideration has $n - m$ degrees of freedom. Admissible variations $\delta\boldsymbol{q}$ have to belong to the tangent space to Q at $\boldsymbol{q} \in \mathrm{Q}$ given by

$$T_q\mathrm{Q} = \{\boldsymbol{v} \in \mathbb{R}^n | \nabla g_l(\boldsymbol{q}) \cdot \boldsymbol{v} = 0,\ 1 \leqslant l \leqslant m\} \tag{5}$$

Remark 2.1. The variational form (2) of the equations of motion is equivalent to Lagrange's equations (of the first kind), which may be linked to the Lagrange-d'Alembert principle

$$\delta \int_{t_0}^{t_N} \left(T(\dot{\boldsymbol{q}}) - \sum_{l=1}^{m} \lambda^l g_l(\boldsymbol{q})\right) dt + \int_{t_0}^{t_N} \delta\boldsymbol{q} \cdot \boldsymbol{F}\, dt = 0 \tag{6}$$

The Lagrange-d'Alembert principle can be viewed as an extension of Hamilton's principle to account for external forcing, see Marsden and Ratiu [18].

Remark 2.2. In the above description $\boldsymbol{F} \in \mathbb{R}^n$ is loosely termed 'external force vector'. In a multibody system formulated in terms of natural coordinates each individual component of F refers to a specific rigid body (see Section 3 for further details) or a specific node of the finite element discretization of a flexible beam or shell component. Thus the action of joint-forces can be represented by components of $\boldsymbol{F}$, although joint-forces are internal forces (or torques) from the multibody system perspective. If the external force components are to represent joint-forces Newton's third (or action-reaction) law has to be obeyed.

3. Rigid body dynamics in terms of skew coordinates

We next present a reformulation of the rotationless formulation of rigid body dynamics (Betsch and Steinmann [3] and Saletan and Cromer [15, Chapter 5]). The original version of the rotationless formulation relies on the assumption of an orthonormal director frame. The orthonormality of the director frame is related to the rigid body assumption and enforced by algebraic constraints. However, the direct discretization of the DAEs typically relaxes the constraints to discrete points in time (see Section 6). Correspondingly, in the discrete setting the orthonormality of the director frame is confined to discrete (or nodal) points in time. This implies that convex combinations of the nodal directors in the discrete setting represent base vectors that are in general neither of unit length nor mutually orthogonal. This deficiency (or more specifically, the discretization error) can be taken into account by introducing skew coordinates from the outset. In particular, the use of skew coordinates turns out to be beneficial to the formulation and consistent numerical discretization of external torques.

In the following we use convected coordinates θ^i to label a material point belonging to the rigid body (Fig. 1). The position of a material point at time t can be described by

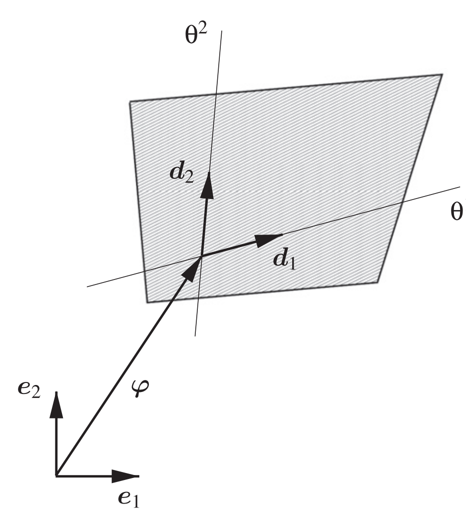

Fig. 1. Planar sketch of the rigid body.