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# Homogenization of a space frame as a thick plate: Application of the Bending-Gradient theory to a beam lattice

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ABSTRACT

The Bending-Gradient theory for thick plates is the extension to heterogeneous plates of Reissner–Mindlin theory originally designed for homogeneous plates. In this paper the Bending-Gradient theory is extended to in-plane periodic structures made of connected beams (space frames) which can be considered macroscopically as a plate. Its application to a square beam lattice reveals that classical Reissner–Mindlin theory cannot properly model such microstructures. Comparisons with exact solutions show that only the Bending-Gradient theory captures second order effects in both deflection and local stress fields.


## 1. Introduction

The classical theory of plates, known also as Kirchhoff–Love plate theory is based on the assumption that the normal to the mid-plane of the plate remains normal after transformation. This theory is also the first order of the asymptotic expansion with respect to the thickness [1]. Thus, it presents a good theoretical justification and was soundly extended to the case of periodic plates [2]. It enables to have a first order estimate of the macroscopic deflection as well as local stress fields. In most applications the first order deflection is accurate enough. However, this theory does not capture the local effect of shear forces on the microstructure because shear forces are one higher-order derivative of the bending moment in equilibrium equations ($Q_\alpha = M_{\alpha\beta,\beta}$).

Because shear forces are part of the macroscopic equilibrium of the plate, their effect is also of great interest for engineers when designing structures. However, modeling properly the action of shear forces is still a controversial issue. Reissner [3] suggested a model for homogeneous plates based on a parabolic distribution of transverse shear stress through the thickness (Reissner–Mindlin theory). This model performs well for homogeneous plates and gives more natural boundary conditions than those of Kirchhoff–Love theory. Thus, it is appreciated by engineers and broadly used in applied mechanics. However, the direct extension of this model to more complex microstructures raised many difficulties. Many suggestions were made for laminated plates [4–9] as well as in-plane periodic plates [10,11], leading to more complex models.

Revisiting the approach from Reissner [3] directly with laminated plates, Lebée and Sab [12,13] showed that the transverse shear static variables which come out when the plate is heterogeneous are not shear forces $Q_\alpha$ but the full gradient of the bending moment $R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma}$. Using conventional variational tools, they derived a new plate theory – called Bending-Gradient theory – which is actually turned into Reissner–Mindlin theory when the plate is homogeneous. This new plate theory is seen by the authors as an extension of Reissner's theory to heterogeneous plates which preserves most of its simplicity. Originally designed for laminated plates, it was also extended to in-plane periodic plates using averaging considerations such as Hill–Mandel principle and successfully applied to sandwich panels [14,15].

In order to give a more comprehensive illustration of the features of this new theory, we extend its homogenization scheme to space frames (Section 2). Space frames are large roofings made of many identical unit-cells. Numerous illustration are given in Buckminster Fuller's achievements. In this work, a space frame is a unit-cell made of connected beams periodically reproduced in a plane and which "from far" can be considered as a plate. Many devices fall into this category: space trusses [16–19], tensegrity, nexorade [20], gridshells [21], lattices, expanded metal, gratings, *etc.*

Let's point out that very few methods exist in the literature for deriving a thick plate macroscopic model when the microstructure is made of structural elements. Clearly, theories for laminates are not suitable for the present microstructure and we need an approach which takes into account the periodicity of the plate. There is also a large literature dedicated to the homogenization of beam lattices (also referred as discrete medias [22,23]. However, these approaches lead only to 3D or in-plane macroscopic models.

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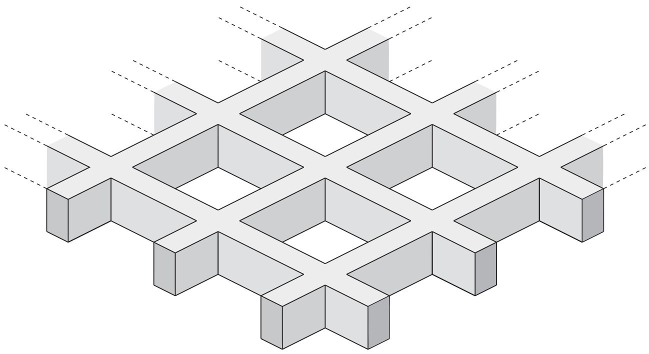

**Fig. 1.** A square beam lattice.

Finally, works from Lewinski [10] and Cecchi and Sab [11] suggest homogenization techniques for a thick plate which apply only to microstructures with the classical Cauchy's continuum. Hence, in this respect the present approach is innovative.

One can argue that from an engineering point of view, the full simulation of the lattice remains affordable in terms of computation and will directly lead to more accurate results. However, because these structures are periodic, their design is often based on the assessment of a single unit-cell. Engineers check if this unit-cell sustains highest loads in critical areas and then, set the design for the entire roof. Having a single unit-cell reproduced many times enables returns to scale and reduces costs. Such an approach requires a good knowledge of local stress fields in the unit-cell generated by macroscopic loadings, independently of the configuration of the plate, which is exactly the purpose of homogenization techniques. Hence these approaches are complementary of full simulations.

In Section 3 we consider a square beam lattice (Fig. 1) and apply both Bending-Gradient and Cecchi and Sab [11] Reissner–Mindlin homogenization scheme. This very simple pattern will enable the derivation of closed-form solutions of the auxiliary problems which are easy to interpret even if the approach can handle 3D geometries. Let us already point out that, because of the patterns symmetries, the Reissner–Mindlin shear forces stiffness $\underset{\sim}{\boldsymbol{F}}$ ($Q_\alpha = F_{\alpha\beta}\gamma_\beta$, where $\gamma_\alpha$ is Reissner–Mindlin shear strain) is necessarily isotropic[1]: $F_{\alpha\beta} = F\delta_{\alpha\beta}$. However, the pattern must be somehow sensitive to the bending orientation.

In order to check this prediction, a comparison with exact solutions of the cylindrical bending of the lattice in two orientations is performed in Section 4. It reveals that only the Bending-Gradient theory is able to capture second order effects both in terms of deflection and local stress fields.

## 2. Homogenization of a periodic space frame as a thick plate

Full details about the Bending-Gradient plate theory are provided in [12–14]. In this section we recall its main features and derive a homogenization scheme dedicated to space frames.

### 2.1. Summary of the Bending-Gradient plate model

We consider a linear elastic plate which mid-plane is the 2D domain $\omega \subset \mathbb{R}^2$. Cartesian coordinates $(x_1, x_2, x_3)$ in the reference frame $(\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \underline{\boldsymbol{e}}_3)$ are used to describe macroscopic fields. At this stage, the microstructure of the plate is not specified.

The membrane stress $N_{\alpha\beta}$, the bending moment $M_{\alpha\beta}$, and shear forces $Q_\alpha$ $(\alpha, \beta, \gamma \ldots = 1, 2)$ are the usual generalized stresses for plates. Moreover, the main feature of the Bending-Gradient theory is the introduction of an additional static unknown: the gradient of the bending moment $R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma}$. The 2D third-order[2] tensor $\underset{\frown}{\boldsymbol{R}}$ complies with the following symmetry: $R_{\alpha\beta\gamma} = R_{\beta\alpha\gamma}$. It is possible to derive shear forces $\underline{\boldsymbol{Q}}$ from $\underset{\frown}{\boldsymbol{R}}$ with: $Q_\alpha = R_{\alpha\beta\beta}$.

The full bending gradient $\underset{\frown}{\boldsymbol{R}}$ has six components (taking into account symmetries of indices) whereas $\underline{\boldsymbol{Q}}$ has two components. Thus, using the full bending gradient as static unknown introduces four additional static unknowns. More precisely: $R_{111}$ and $R_{222}$ are respectively the cylindrical bending part of shear forces $Q_1$ and $Q_2$, $R_{121}$ and $R_{122}$ are respectively the torsion part of these shear forces and $R_{112}$ and $R_{221}$ are linked to strictly self-equilibrated stresses. Equilibrium equations and stress boundary conditions are detailed in Appendix A. They are very similar to those of Reissner–Mindlin theory where $Q_\alpha = M_{\alpha\beta,\beta}$ is replaced by $R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma}$.

The main difference between Reissner–Mindlin and Bending-Gradient plate theories is that the latter enables the distinction between each component of the gradient of the bending moment whereas they are mixed into shear forces with Reissner–Mindlin theory.

Generalized stresses $N_{\alpha\beta}$, $M_{\alpha\beta}$ and $R_{\alpha\beta\gamma}$ work respectively with the dual strain variables: $e_{\alpha\beta}$, the conventional membrane strain, $\chi_{\alpha\beta}$ the curvature and $\Gamma_{\alpha\beta\gamma}$ the third order tensor related to generalized shear strains. These strain fields must comply with the compatibility conditions and boundary conditions detailed in Appendix A.

Finally, assuming uncoupling between $(\underset{\sim}{\boldsymbol{N}}, \underset{\sim}{\boldsymbol{M}})$ and $\underset{\frown}{\boldsymbol{R}}$ (see [12,14], the Bending-Gradient plate constitutive equations are written as:

$$\underset{\sim}{\boldsymbol{N}} = \underset{\approx}{\boldsymbol{A}} : \underset{\sim}{\boldsymbol{e}} + \underset{\approx}{\boldsymbol{B}} : \underset{\sim}{\boldsymbol{\chi}} \tag{1a}$$

$$\underset{\sim}{\boldsymbol{M}} = {}^T\underset{\approx}{\boldsymbol{B}} : \underset{\sim}{\boldsymbol{e}} + \underset{\approx}{\boldsymbol{D}} : \underset{\sim}{\boldsymbol{\chi}} \tag{1b}$$

$$\underset{\frown}{\boldsymbol{\Gamma}} = \underset{\frown\frown}{\boldsymbol{f}} \,\vdots\, \underset{\frown}{\boldsymbol{R}} \tag{1c}$$

where $\left(\underset{\approx}{\boldsymbol{A}}, \underset{\approx}{\boldsymbol{B}}, \underset{\approx}{\boldsymbol{D}}\right)$ are conventional Kirchhoff–Love fourth order stiffness tensors and " : " is the double contraction product. The generalized shear compliance tensor $\underset{\frown\frown}{\boldsymbol{f}}$ is a sixth order tensor[3] and " $\vdots$ " denotes a triple contraction product: $\underset{\frown\frown}{\boldsymbol{f}} \vdots \underset{\frown}{\boldsymbol{R}} = (f_{\alpha\beta\gamma\delta\epsilon\zeta} R_{\zeta\epsilon\delta})$.

In some cases, the Bending-Gradient is turned into a Reissner–Mindlin plate model. This is the case for homogeneous plates. In order to estimate the difference between both plate models we defined the isotropic projection of the Bending-Gradient stress energy density on a Reissner–Mindlin one in [12]. According to this projection, the Reissner–Mindlin part of $\underset{\frown\frown}{\boldsymbol{f}}$ is:

$$\underset{\frown\frown}{\boldsymbol{f}}^{RM} = \left(\frac{2}{3} \underset{\approx}{\boldsymbol{i}} \cdot \underset{\approx}{\boldsymbol{i}}\right) \vdots \underset{\frown\frown}{\boldsymbol{f}} \vdots \left(\frac{2}{3} \underset{\approx}{\boldsymbol{i}} \cdot \underset{\approx}{\boldsymbol{i}}\right) \tag{2}$$

where $\underset{\approx}{\boldsymbol{i}}$ is the identity for in-plane elasticity tensors $(i_{\alpha\beta\gamma\delta} = \frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}))$ and $\underset{\approx}{\boldsymbol{i}} \cdot \underset{\approx}{\boldsymbol{i}} = (i_{\alpha\beta\gamma\eta} i_{\eta\delta\epsilon\zeta})$ is a sixth order tensor. Consequently, we suggested the following relative distance between the Bending-Gradient and the Reissner–Mindlin stress energy densities:

$$\Delta^{RM/BG} = \frac{\| \underset{\frown\frown}{\boldsymbol{f}} - \underset{\frown\frown}{\boldsymbol{f}}^{RM} \|}{\| \underset{\frown\frown}{\boldsymbol{f}} \|}, \quad \text{where } \| \underset{\frown\frown}{\boldsymbol{f}} \| = \sqrt{f_{\alpha\beta\gamma\delta\epsilon\zeta} f_{\alpha\beta\gamma\delta\epsilon\zeta}} \tag{3}$$

[1] Two orthogonal direction of orthotropy and invariance under 90° in-plane rotation.

[2] Vectors and higher-order tensors are boldfaced and different underlinings are used for each order: vectors are straight underlined, $\underline{\boldsymbol{u}}$. Second order tensors are underlined with a tilde: $\underset{\sim}{\boldsymbol{M}}$. Third order tensors are underlined with a parenthesis: $\underset{\frown}{\boldsymbol{R}}$. Fourth order tensors are doubly underlined with a tilde: $\underset{\approx}{\boldsymbol{D}}$. Sixth order tensors are doubly underlined with a parenthesis: $\underset{\frown\frown}{\boldsymbol{f}}$.

[3] $f_{\alpha\beta\gamma\delta\epsilon\zeta}$ follows major symmetry: $f_{\alpha\beta\gamma\delta\epsilon\zeta} = f_{\zeta\epsilon\delta\gamma\beta\alpha}$ and minor symmetry $f_{\alpha\beta\gamma\delta\epsilon\zeta} = f_{\beta\alpha\gamma\delta\epsilon\zeta}$. Thus there are only 21 independent components.