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Regularized formulation of potential field gradients in singular boundary method

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A R T I C L E I N F O

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A B S T R A C T

The main difficulty of the singular boundary method (SBM) is the calculation of the origin intensity factors introduced to remove the singularity of fundamental solution. This work presents an extension of the previous SBM formation. The new contribution of the present method is that the origin intensity factors for potential field gradients are derived, so that the boundary derivatives of the potential at any direction, not limited to the normal derivative, can now be calculated. The accuracy and efficiency of the present method are verified through several numerical examples. All the numerical results demonstrate that the method performs well in comparisons with the exact solution, the method of fundamental solutions (MFS) and boundary element method (BEM).

1. Introduction

During the past decades, many researchers have paid attention to the meshless methods and their application [1–11] owing to no domain and boundary elements. Among the meshless methods, we focus on a variety of boundary-type methods, such as the method of fundamental solutions (MFS) [3,12–14], regularized meshless method (RMM) [15,16], modified method of fundamental solutions (MMFS) [5], boundary distributed source (BDS) method [17,18], and singular boundary method (SBM) [7,19,20].

The MFS is very simple and easy to program, which was provided to eliminate the drawbacks of the boundary element method (BEM) [21–23]. The method distributed the source points on a fictitious boundary to avoid the singularities of the fundamental solutions. So far, there are various techniques [24–26] to effectively select the sub-optimal fictitious boundary. But nevertheless it is still not an easy work for the MFS solution of multiply-connected and complex domain problems. To remove the fictitious boundary in the MFS, Young et al. proposed the RMM [15]. Compared with the MFS, the RMM interpolation matrix has better mathematical properties even with a larger number of boundary nodes. But unfortunately, the RMM formulations are derived based on the double-layer potentials which have hyper-singularity at the origin. Besides, Sarler [5] proposed the MMFS by employing the singular-layer potential to solve potential flow problems. The approach is stable and no requirement of a fictitious boundary. However, it is complex to evaluate the diagonal elements of interpolation matrix.

In this study, we focus on the SBM recently proposed by Chen [7]. The approach introduces origin intensity factors to isolate the singularity of the fundamental solutions. Therefore, the key problem of the SBM is how to calculate the origin intensify factors. Once these factors are determined, the solution of the problem of interest can be approximated by a linear combination of fundamental solutions with source points located directly on the physical boundary. This is dramatically different from the MFS requiring the perplexing fictitious boundary to distribute the source points. Recently, the SBM has since been successfully applied to various engineering problems [19,27–33].

This paper is an extension of the previous published SBM where the origin intensity factors for Dirichlet and Neumann boundary conditions are derived. Herein, we propose a new SBM formulation to calculate the origin intensity factors for potential field gradients, so that the boundary derivatives of the potential at any direction, not limited to the normal derivative, can now be evaluated. Moreover, with combining the origin intensity factors for potential gradients, origin intensity factors for Neumann boundary conditions can be obtained indirectly. A brief outline of the paper is as follows. Section 2 reviews the previous SBM formulations for the potential problems. Section 3 derives the origin intensity factors for potential field gradients. Section 4 provides four benchmark test problems to validate the computational code and assess the performances of the proposed SBM scheme. Finally, Section 5 concludes the paper.

2. The SBM for potential field theory

We consider two dimensional (2D) Laplace equation governing potential problems [34]

$$\nabla^2 u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega. \tag{1}$$

subject to the Dirichlet boundary condition

$$u(\mathbf{x}) = \bar{u}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_D, \tag{2}$$

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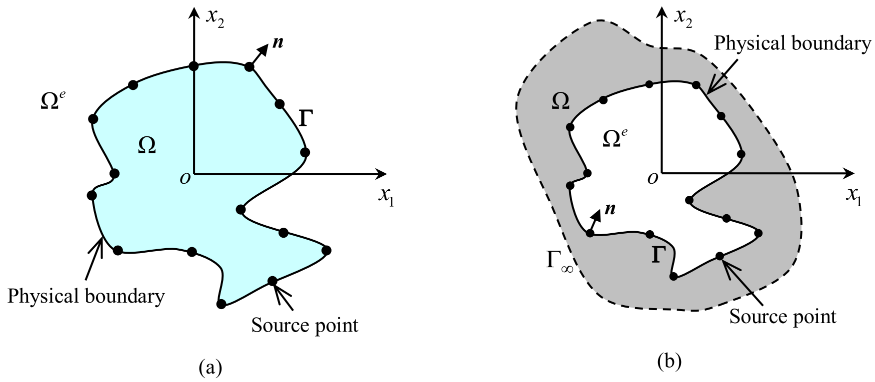

Fig. 1. The source point distributions: (a) the SBM for interior problems, (b) the SBM for exterior problems.

or the Neumann boundary condition

$$q(\mathbf{x}) = \frac{\partial u(\mathbf{x})}{\partial \mathbf{n}} = \bar{q}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_N, \tag{3}$$

where u is the potential field, $\Gamma = \Gamma_D + \Gamma_N$ the whole boundary of the domain Ω, $\boldsymbol{n}$ the outward normal vector, $\bar{u}, \bar{q}$ given values on the boundary. The domain Ω, the boundary Γ, and normal vector $\mathbf{n}$ can be found in Fig. 1. For exterior problems, the solution $u(\mathbf{x})$ also has to satisfy a condition at infinity [35]

$$\lim_{\|\mathbf{x}\|_2 \to \infty} u(\mathbf{x}) = \hat{c}, \tag{4}$$

in which $\|\mathbf{x}\|_2$ represents the Euclidean distance, and $\hat{c}$ is a finite constant.

For the SBM, the source and collocation points are the same set of the boundary nodes placed on the physical boundary, as seen in Fig. 1. The SBM formulations for interior potential problems are given by [19]

$$u(\mathbf{x}^i) = \sum_{j=1, j\neq i}^{N} \alpha^j G^{(\mathrm{I})}(\mathbf{x}^i, \mathbf{s}^j) + \alpha^i u_{ii}, \quad \mathbf{x}^i \in \Gamma_D, \mathbf{s}^j \in \Gamma, \tag{5}$$

$$q(\mathbf{x}^i) = \sum_{j=1, j\neq i}^{N} \alpha^j \frac{\partial G^{(\mathrm{I})}(\mathbf{x}^i, \mathbf{s}^j)}{\partial \mathbf{n}(\mathbf{x}^i)} + \alpha^i q_{ii}, \quad \mathbf{x}^i \in \Gamma_N, \mathbf{s}^j \in \Gamma, \tag{6}$$

where $\mathbf{x}^i$ is the ith collocation point, $\mathbf{s}^j$ the jth source point, $\alpha^j (j = 1, \ldots, N)$ the jth unknown coefficient, N the number of source points, and $G^{(\mathrm{I})}(\mathbf{x}^i, \mathbf{s}^j)$ the fundamental solution

$$G^{(\mathrm{I})}(\mathbf{x}^i, \mathbf{s}^j) = \frac{1}{2\pi} \ln \frac{1}{\|\mathbf{x}^i - \mathbf{s}^j\|_2}, \tag{7}$$

in which the superscript (I) of G denotes the interior problems. u_{ii} and q_{ii} are origin intensity factors

$$u_{ii} = \frac{1}{\phi^i}\left(\bar{u}(\mathbf{x}^i) - \sum_{j=1, j\neq i}^{N} \phi^j G^{(\mathrm{I})}(\mathbf{x}^i, \mathbf{s}^j) - c\right), \tag{8}$$

$$q_{ii} = -\sum_{j=1, j\neq i}^{N} \frac{\partial G^{(\mathrm{I})}(\mathbf{x}^i, \mathbf{s}^j)}{\partial \mathbf{n}(\mathbf{s}^j)}. \tag{9}$$

In Eq. (8), $\bar{u}(\mathbf{x})$ is an arbitrary known particular solution; $\phi^i (i = 1, 2, \ldots, N)$ and c can be carried out with an inverse interpolation technique (IIT), and the details can be found in Ref. [19].

A linear system of equations is formed by using Eqs. (5) and (6), and then the unknown coefficients $\alpha^j (j = 1, \cdots, N)$ can be determined. We use a direct solver: Gaussian elimination method to solve this mentioned linear system. The potential inside the domain can be finally evaluated by the equation as follows:

$$u(\mathbf{x}) = \sum_{j=1}^{N} \alpha^j G^{(I)}(\mathbf{x}, \mathbf{s}^j), \quad \mathbf{x} \in \Omega, \mathbf{s}^j \in \Gamma. \tag{10}$$

Similarly, the representation of the solution for exterior problems can be expressed as [36]

$$u(\mathbf{x}^i) = \sum_{j=1, j\neq i}^{N} \alpha^j G^{(\mathrm{E})}(\mathbf{x}^i, \mathbf{s}^j) + \alpha^i u_{ii} + \hat{c}, \quad \mathbf{x}^i \in \Gamma_D, \mathbf{s}^j \in \Gamma, \tag{11}$$

$$q(\mathbf{x}^i) = \sum_{j=1, j\neq i}^{N} \alpha^j \frac{\partial G^{(\mathrm{E})}(\mathbf{x}^i, \mathbf{s}^j)}{\partial \mathbf{n}(\mathbf{x}^i)} + \alpha^i q_{ii}, \quad \mathbf{x}^i \in \Gamma_N, \mathbf{s}^j \in \Gamma, \tag{12}$$

where the superscript (E) of G denotes the exterior problems, and u_{ii} and q_{ii} are origin intensity factors [36] for exterior problems. Unknown $\alpha^j (j = 1, \ldots, N)$ and $\hat{c}$ are determined by using Eqs. (11) and (12) with constraint condition $\sum_{j=1}^{N} \alpha^j = 0$. It should be noted that $G^{(E)}$ has the same expression (7) of $G^{(I)}$ and the superscripts (I) and (E) of G are used to distinguish the different integral identities of the fundamental solution (see Eqs. A.3 and A.4 in Appendix A). Based on these two superscripts, we will more clearly derive the SBM formulations of potential field gradients for the interior and exterior problems in the following section.

Compared with the MFS and BEM, the SBM has the following pros (1)–(3) and cons (4), (5):

1. It avoids artificial boundary encountered in the MFS;
2. It has good stability;
3. It is mathematically simple and easy to program compared with the BEM;
4. A better strategy is highly desired to evaluate the origin intensity factor to remove the singularities of the fundamental solution;
5. As a recent technique, the SBM remains immature to be tested to a variety of large-scale complex problems as compared with well-established BEM.

3. Regularized SBM formulation for potential field gradients

In this section, we propose a new SBM formulation to calculate the origin intensity factors for potential field gradients, so that the boundary derivatives of the potential at any direction, not limited to the normal derivative, can be calculated.

3.1. Interior problems

Based on Eq. (10), the potential field gradient at collocation point $\mathbf{x}^i (\neq \mathbf{s}^j)$ can be approximated using a linear combination of the fundamental solution derivatives

$$u_{,k}(\mathbf{x}^i) = \frac{\partial u(\mathbf{x}^i)}{\partial x_k} = \sum_{j=1}^{N} \alpha^j \frac{\partial G^{(\mathrm{I})}(\mathbf{x}^i, \mathbf{s}^j)}{\partial x_k}, \quad k = 1, 2, \tag{13}$$