Engineering Analysis with Boundary Elements 95 (2018) 187–199

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Numerical simulation of nonlinear coupled Burgers' equation through meshless radial point interpolation method

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A R T I C L E I N F O

Keywords:
Spectral meshless radial point interpolation (SMRPI) method
Radial basis function
Nonlinear coupled Burgers' equation

A B S T R A C T

In present paper, the spectral meshless radial point interpolation (SMRPI) technique is applied to the solution of nonlinear coupled Burgers' equation in two dimensions. Firstly, we obtain a time discrete scheme by approximating time derivative via a finite difference formula, then we use the SMRPI approach to approximate the spatial derivatives. This method is based on a combination of meshless methods and spectral collocation techniques. The point interpolation method with the help of radial basis functions is used to construct shape functions which act as basis functions in the frame of SMRPI. In the current work, the thin plate splines (TPS) are used as the basis functions and in order to eliminate the nonlinearity, a simple predictor-corrector (P-C) scheme is performed. The aim of this paper is to show that the SMRPI method is suitable for the treatment of nonlinear coupled Burgers' equation. With regard to test problems that have not exact solutions, we consider two strategies for checking the stability of time difference scheme and for survey the convergence of the fully discrete scheme. The results of numerical experiments confirm the accuracy and efficiency of the presented scheme.

1. Introduction

As mentioned in [1], Mathematical models of basic flow equations describing unsteady transport problems consist of system of nonlinear parabolic and hyperbolic PDEs. The coupled Burgers' equations [2] form an important class of such PDEs. This class is related to a large number of physical problems such as the phenomena of turbulence and supersonic flow, flow of a shock wave traveling in a viscous fluid, sedimentation of two kinds of particles in fluid suspensions under the effect of gravity, acoustic transmission, traffic and aerofoil flow theory, as well as a prerequisite to the Navier-Stokes equations [2–6].

The present paper considers the nonlinear coupled Burgers' equation in two dimensions as follows:

$$\begin{cases} \dfrac{\partial u(\mathbf{x},t)}{\partial t} + u(\mathbf{x},t)\dfrac{\partial u(\mathbf{x},t)}{\partial x} + v(\mathbf{x},t)\dfrac{\partial u(\mathbf{x},t)}{\partial y} = \alpha\Delta u(\mathbf{x},t), \\ \dfrac{\partial v(\mathbf{x},t)}{\partial t} + u(\mathbf{x},t)\dfrac{\partial v(\mathbf{x},t)}{\partial x} + v(\mathbf{x},t)\dfrac{\partial v(\mathbf{x},t)}{\partial y} = \beta\Delta v(\mathbf{x},t), \\ \mathbf{x}=(x,y)\in\Omega\subset\mathbb{R}^2, \qquad t\in[0,T], \end{cases} \tag{1}$$

with initial conditions

$$u(\mathbf{x},0)=u_0(\mathbf{x}), \qquad v(\mathbf{x},0)=v_0(\mathbf{x}), \qquad \mathbf{x}\in\Omega, \tag{2}$$

and Dirichlet boundary conditions

$$u(\mathbf{x},t)=h_1(\mathbf{x},t), \qquad v(\mathbf{x},t)=h_2(\mathbf{x},t), \qquad \mathbf{x}\in\partial\Omega, t>0, \tag{3}$$

where α and β are positive viscosity parameters which correspond to an inverse of Reynolds number Re (if $\alpha=\beta$, then $\alpha=\beta=1/Re$), $u(\mathbf{x},t)$ and $v(\mathbf{x},t)$ are the velocity components in two dimension; u_0, v_0, h_1 and h_2 are known functions; $\partial u/\partial t$ is unsteady term, $u\partial u/\partial x$ is the nonlinear convection term, Δ is the Laplacian differential operator and $\alpha\Delta u(\mathbf{x},t)$ is diffusion term.

Exact solution for the special cases of the two dimensional coupled Burgers' equations is given by Fletcher [7] using Hopf–Cole transformation. Much work has been done on developing numerical methods for solving the numerical solution of nonlinear coupled Burgers' equation. Some examples are described below. The authors of [8] studied nonlinear Burgers' equation by polynomial differential quadrature method. The authors of [1] presented a meshless local radial basis functions collocation method (LRBFCM) to the numerical solution of the transient nonlinear coupled Burgers' equations with Dirichlet and mixed boundary conditions. Mohammadi et al. [9] employed Galerkin-reproducing kernel method for solving the 2D nonlinear coupled Burgers' equations having Dirichlet and mixed boundary conditions. Shukla et al. [10] studied the numerical solution of two dimensional nonlinear coupled viscous Burger equations through modified cubic B-spline differential quadrature method. Tamsir et al. [11] developed a new differential quadrature method "exponential modified cubic B-spline differential quadrature method" on one and two dimensional nonlinear Burgers' equations having Dirichlet boundary conditions. Stability analysis of the proposed algorithm is also done by using matrix stability analysis method. The authors of [12] introduced two new modified fourth-order exponential time differencing Runge–Kutta (ETDRK) schemes in combination with a global fourth-order compact finite difference scheme (in space) for

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https://doi.org/10.1016/j.enganabound.2018.07.014
Received 31 January 2018; Received in revised form 3 June 2018; Accepted 12 July 2018
0955-7997/

direct integration of nonlinear coupled viscous Burgers' equations in their original form with out using any transformations or linearization techniques.

The main shortcoming of mesh-based methods such as the finite element method (FEM) [13], the finite volume method (FVM) [14] and the boundary element method (BEM) [15] is that these numerical methods rely on meshes or elements. In the two last decades, in order to overcome the mentioned difficulties some techniques so-called meshless methods have been proposed [16,17]. A brief review of the meshless method has been studied in [18].

In spite of great benefits in using the meshless weak form methods, there are some limitations. For example, the complicated nature of the non-polynomial shape functions may be computationally expensive to implement in a numerical integration scheme. On the other hand, some methods such as those that are based on moving least squares (MLS) and RBFs, need to determine a shape parameter which plays the important role in the accuracy of the methods. Furthermore, the resultant linear systems might be ill-conditioned and to overcome this defect, some regularization methods are needed. In the meshless method based on strong form, such as Kansa's method, this RBF collocation approach is inherently meshless, easy-to-program, and mathematically very simple to learn, but its fundamental flaw is un-stability because of the use of the global strong form. To overcome these shortages, we propose a new spectral meshless radial point interpolation (SMRPI) method which is based on meshless radial point interpolation and spectral collocation techniques [19–23]. In the SMRPI method, the point interpolation method by the help of radial basis functions is proposed to construct shape functions which have Kronecker delta function property and are used as basis functions in the frame of the SMRPI. Based on the spectral methods, evaluation of high-order derivatives of given differential equation is easy by constructing and using operational matrices. The SMRPI method does not require any kind of integration locally over small quadrature domains nor regularization techniques. Therefore, the computational cost of the SMRPI method is less expensive.

The outline of this paper is as follows: The time discrete scheme for implementation of the SMRPI is given in Section 2. In Section 3, we introduce the spectral meshless radial point interpolation scheme briefly and shape functions are constructed. The implementation of the SMRPI for time discrete equation is given in Section 4. In Section 5, to show the accuracy and efficiency of the proposed method some numerical results, are investigated. Finally a conclusion is given in Section 6.

2. Time discrete scheme

Let us define

$$t_k = k\delta t, \qquad k = 0, 1, \dots, M,$$

where $\delta t = T/M$ is the step size of time variable. In this section, we discretize the time variable using forward finite difference relation for the first order derivatives on time variable with the Crank–Nicolson scheme, appropriately, as follows

$$\frac{\partial u(\mathbf{x}, t)}{\partial t} \cong \frac{u^{k+1}(\mathbf{x}) - u^k(\mathbf{x})}{\delta t}, \tag{4}$$

$$\Delta u(\mathbf{x}, t) \cong \frac{1}{2}\left(\Delta u^{k+1}(\mathbf{x}) + \Delta u^k(\mathbf{x})\right), \tag{5}$$

where $u^{k+1}(\mathbf{x})$ is approximate solution at the point $(\mathbf{x}, t_{k+1})$. Applying the above approximation and impose them to the original Eq. (1), we are conducted to the following time discrete equation:

$$\begin{cases} \dfrac{u^{k+1}(\mathbf{x}) - u^k(\mathbf{x})}{\delta t} + u^k(\mathbf{x})\dfrac{\partial u^k(\mathbf{x})}{\partial x} + v^k(\mathbf{x})\dfrac{\partial u^k(\mathbf{x})}{\partial y} = \dfrac{\alpha}{2}\left(\Delta u^{k+1}(\mathbf{x}) + \Delta u^k(\mathbf{x})\right), \\ \dfrac{v^{k+1}(\mathbf{x}) - v^k(\mathbf{x})}{\delta t} + u^k(\mathbf{x})\dfrac{\partial v^k(\mathbf{x})}{\partial x} + v^k(\mathbf{x})\dfrac{\partial v^k(\mathbf{x})}{\partial y} = \dfrac{\beta}{2}\left(\Delta v^{k+1}(\mathbf{x}) + \Delta v^k(\mathbf{x})\right). \end{cases} \tag{6}$$

Then, Eq. (6) can be rewritten as

$$\begin{cases} \lambda u^{k+1}(\mathbf{x}) - \alpha\Delta u^{k+1}(\mathbf{x}) = \lambda u^k(\mathbf{x}) + \alpha\Delta u^k(\mathbf{x}) - 2u^k(\mathbf{x})\dfrac{\partial u^k(\mathbf{x})}{\partial x} - 2v^k(\mathbf{x})\dfrac{\partial u^k(\mathbf{x})}{\partial y}, \\ \lambda v^{k+1}(\mathbf{x}) - \beta\Delta v^{k+1}(\mathbf{x}) = \lambda v^k(\mathbf{x}) + \beta\Delta v^k(\mathbf{x}) - 2u^k(\mathbf{x})\dfrac{\partial v^k(\mathbf{x})}{\partial x} - 2v^k(\mathbf{x})\dfrac{\partial v^k(\mathbf{x})}{\partial y}, \end{cases} \tag{7}$$

where $\lambda = 2/\delta t$.

3. Radial point interpolation

In this section our purpose is to obtain shape function in SMRPI approach. Since we use the radial basis function of conditionally positive definite to build the shape functions, we remember the following definition and theorem from [24].

Definition 1. A continuous function $\varphi : \mathbb{R}^d \to \mathbb{C}$ is said to be conditionally positive semi-definite of order m (i.e. to have conditional positive deniteness of order m) if, for all $N \in \mathbb{N}$, all pairwise distinct centers $x_1, \dots, x_N \in \mathbb{R}^d$, and all $\mathbf{c} = [c_1, \dots, c_N]^{tr} \in \mathbb{C}^N$ satisfying

$$\sum_{j=1}^{N} c_j p(x_j) = 0,$$

for all complex-valued polynomials of degree less than m, the quadratic form

$$\sum_{j=1}^{N}\sum_{k=1}^{N} c_j \bar{c}_k \varphi(x_j - x_k)$$

is nonnegative. φ is said to be conditionally positive definite of order m if the quadratic form is positive, unless $\mathbf{c}$ is zero.

Theorem 1. ***(Micchelli)*** *Suppose that $\varphi \in C[0, \infty) \cap C^\infty(0, \infty)$ is given. Then the function $\phi = \varphi(\|.\|_2^2)$ is conditionally positive semi-definite of order $m \in \mathbb{N}_0$ on every $\mathbb{R}^d$ if and only if $(-1)^m\varphi^{(m)}$ is completely monotone on* $(0, \infty)$.

Now, one can find many conditionally positive definite functions by using this theorem. For an example, the thin-plate or surface splines $\varphi(r) = (-1)^{k+1}r^{2k}\log(r)$ are conditionally positive definite of order $m = k+1$ on every $\mathbb{R}^d$. Consider a continuous function $u(\mathbf{x})$ defined in a domain $\Omega \subset \mathbb{R}^2$, which is represented by a set of field nodes. The $u(\mathbf{x})$ at a point of interest $\mathbf{x}$ is approximated in the form of

$$u(\mathbf{x}) = \sum_{i=1}^{n} R_i(\mathbf{x})a_i + \sum_{j=1}^{np} P_j(\mathbf{x})b_j = \mathbf{R}^{tr}(\mathbf{x})\mathbf{a} + \mathbf{P}^{tr}(\mathbf{x})\mathbf{b}, \tag{8}$$

where $R_i(\mathbf{x})$ is a radial basis function (RBF), n is the number of RBFs, $P_j(\mathbf{x})$ is monomial in the space coordinate $\mathbf{x}$, and np is the number of polynomial basis functions. It should be noticed that the additional polynomials are not necessary if the RBF is strictly positive definite. Thus, when the TPS is used for interpolation, the polynomial terms should be employed to avoid the singularity. Coefficients a_i and b_j are unknown which should be determined. In order to determine a_i and b_j in Eq. (8), a support domain is formed for the point of interest at $\mathbf{x}$, and n field nodes are included in the support domain (support domain is usually a disk with radius r_s). Coefficients a_i and b_j can be determined by enforcing Eq. (8) to be satisfied at these n nodes surrounding the point of interest $\mathbf{x}$. Therefore, by the idea of interpolation, Eq. (8) is converted to the following form:

$$u(\mathbf{x}) = \mathbf{\Phi}^{tr}(\mathbf{x})\mathbf{U}_s = \sum_{i=1}^{n} \phi_i(\mathbf{x})u_i. \tag{9}$$

It is significant that the RPIM shape functions have the Kronecker delta function property, that is

$$\phi_i(\mathbf{x}_j) = \begin{cases} 1, & i = j, \quad j = 1, 2, \dots, n, \\ 0, & i \neq j, \quad i, j = 1, 2, \dots, n \end{cases}. \tag{10}$$