Engineering Analysis with Boundary Elements 95 (2018) 238–247

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

A novel green element method by mixing the idea of the finite difference method

Xiang Rao[a,*], Linsong Cheng[a], Renyi Cao[a], Jun Jiang[a], Ning Li[a], Sidong Fang[b], Pin Jia[a], Lizhun Wang[a]

[a] *China University of Petroleum, Beijing, China*
[b] *Research Institute of Petroleum Exploration & Development, SINOPEC, Beijing, China*

ARTICLE INFO

Keywords:
Green element method
Finite difference method
Three-order accuracy

ABSTRACT

This paper proposes a novel green element method (GEM) by mixing the idea of finite difference method (FDM). In the novel method, We come to the original formula when boundary integral equation is applied to an element, and use difference quotient of the central nodal value on two sides of the shared edge of adjoining elements to approximate the boundary integration $\int_{\Gamma} G\nabla p \cdot \mathbf{n}ds$. This treatment is similar to FDM, and the integral operator relevant to element size controls the estimated error. The novel GEM makes the numerical solution correspond to the actual physical meaning, and the coefficient matrix of the global matrix is a banded sparse matrix with larger bandwidth than previous GEMs. Meanwhile, the instability of the original GEM is illuminated. We have proven it by theoretical error analysis and five numerical examples that, the accuracy of the novel GEM is three-order higher than the original GEM, and the novel GEM has a good convergence and stability, which is the property that the original GEM does not have.

Indeed, the novel GEM proposed in this paper is essentially a new numerical method mixed with the idea of boundary element method (BEM), finite element method (FEM), and FDM. In contrast with BEM, FEM, FDM and previous GEM, the characteristics of our novel GEM include:

(i) Compared with FEM and FDM, the novel GEM has the accuracy of BEM and can better accord with material balance.

(ii) Compared with BEM, the novel GEM can solve nonlinear problems with heterogeneous media, which are hard to be handled by BEM.

(iii) Compared with previous GEMs, the novel GEM has a three-order accuracy, and has a better convergence that the calculation error can be well controlled by the element size.

1. Introduction

GEM is a promising technique to solve many engineering problems controlled by second order partial differential equations, such as steady flow, heat transfer, nonlinear flow in porous media, etc. GEM is based on boundary integral equation, which is identical to BEM. The essential difference between GEM and BEM lies on that, for GEM the computational domain is divided into a finite number of elements, and boundary integral equation is applied to each element, which is similar to FEM.

Taigbenu [1–4] proposed the original GEM, in which the computational domain was divided by polygons and polygons vertices were used as the solution nodes. In original GEM, the normal flux at each internal node was approximated by differentiating the pressure, and the pressure was obtained by nodal values of the pressure and base functions. Therefore, overall accuracy was reduced. Archer [7–8] used overhauser interpolation functions to reduce the effect of the approximation of the normal flux in original GEM, as a result, accuracy was improved. However, Archer implemented this approach only on rectangular grids and highlighted the problems of using it when the source node was on an external boundary. Lorinczi P [10–13] and Pecher et al. [9] proposed a Flux-Vector-Based GEM also aiming to improve accuracy, and then applied the method to some problems of rectangular and triangular grids. By the method, there are three freedoms in each internal node in two-dimensional domain, and the flux vector and pressure were simultaneously solved in over-determined system of equations, and accuracy was improved to a second-order. Taigbenu [5] proposed a flux-correct GEM, in which he added additional equations of flux term and made pressure and flux simultaneously solved. Taigbenu [6] returned to the original GEM and approximated the interface fluxes in terms of the primary variable, which was allowed to vary as a second order degree polynomial of the spatial dimensions. Rao et al. [14] proposed a GEM based on two sets

* Corresponding author.
E-mail addresses: raoxiang0103@163.com (X. Rao), lscheng@cup.edu.cn (L. Cheng).

https://doi.org/10.1016/j.enganabound.2018.07.015
Received 16 May 2018; Received in revised form 2 July 2018; Accepted 26 July 2018
0955-7997/

of nodes including pressure nodes and flux nodes. The pressure nodes contained all polygon vertices and represented the value of pressure. Based on the fact that the algebraic sum of the normal fluxes at flux nodes on shared edge is zero, the method obtained the global equation systems and actually improved the accuracy. The method, however, was not valid in rectangular grid, since the coefficient matrix of the global equations was either singular or morbid.

This paper is structured along the following lines. In Section 2, we propose a novel GEM by mixing the idea of FDM. In Section 3, the instability of the original GEM is illuminated. In Section 4, five numerical examples of several flow problems are implemented to compare the accuracy and other properties of the novel GEM and original GEM. Finally comes the conclusions in Section 4.

2. The novel GEM formula

We here summarize and analyze the treatment of the flux term in previous GEMs.

(i) In original GEM, flux was approximated by the differential of the potential function expression in Eq. (1). (linear combination of nodal value and base function of interpolation) in the element, which was actually using the linear combination of the derived function of base function and nodal value of the potential function to approximately represent the flux term. It was evident that such treatment may cause huge error, and there were two reasons contributing to such error, (1) the interpolation function was usually taken as linear polynomial whose derived function was a constant with limited accuracy, (2) the potential function in the element was an approximate expression and the differential operator was an unbounded operator, thus error of the estimated flux may be enlarged. Therefore, the original GEM was likely to be lack of a good convergence and stability and high accuracy., which will be introduced in Section 3 in detail.

$$q_n = -K \sum_{i=1}^{N_e} \frac{\partial \phi_i}{\partial n} P_i \tag{1}$$

(ii) In modified flux vector method, normal flux nodes were set on vertices of the polygon element, which utilized the continuity of the component of the flux vector in calculation domain but increased the freedom degree of the nodes. For flux-correct GEM proposed by Taigbenu [5], there was no strict proof to the equations added to the flux term on the nodes.

(iii) Rao et al. [14] proposed a modified GEM based on two sets of nodes and set flux nodes on midpoints of the polygon edges. The algebraic sum of the normal fluxes on shared edge of the adjoining elements equaled to zero, therefore the local equation could be coupled and solved. As a result, the flux term was more consistent with the actual physical significance and the numerical solution was more continuous on shared edge of the adjoining elements, and the calculation efficiency did not be decreased. In the case of 2D rectangular element for Rao's method, however, the coefficient matrix was singular or morbid. Therefore, it could not be applied to rectangular grid.

Based on above analysis, to make flux term consistent with the actual physical significance and numerical solution continuous on shared edge of the adjoining elements, we mix the idea of FDM to process the integration $\int_\Gamma G\nabla p \cdot \mathbf{n} ds$ (where $-k\nabla p \cdot \mathbf{n}$ is called the flux term). The integral operator relevant to element size can also control the estimation error, from which we can infer that error of the new method will decrease to 0 as the element size decreases. Therefore, the novel GEM we propose has a good convergence. The specific details are shown as follows.

As we all know, all previous GEMs are aiming to make numerical solutions of the following second-order partial differential equation, which typically describes the transient flow in heterogeneous media or heat transfer. Notions of the transient flow are used to present the method, such as pressure, flux, etc.

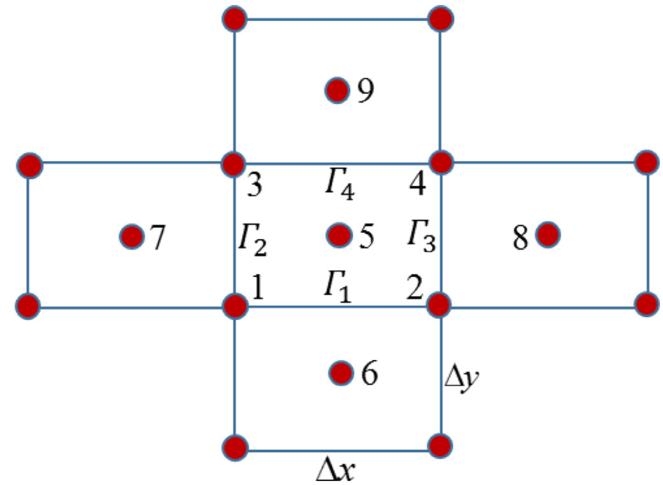

Fig. 1. The sketch of 2D grids.

$$\nabla \cdot (K\nabla p) = c\frac{\partial p}{\partial t} + f \tag{2}$$

where p is the pressure over the computational domain, f is the distribution of internal source strengths, K and c represent the properties of media, which is a function of spatial location in heterogeneous media.

The boundary integral equation derived from Eq. (2) is shown as follows

$$-\lambda p_i + \int_\Gamma \left(p\nabla G \cdot \mathbf{n} + G\frac{q}{K}\right) ds + \iint_\Lambda G\left[-\nabla\psi \cdot \nabla p + \sigma\frac{\partial p}{\partial t} + \nu f\right] dA = 0 \tag{3}$$

where $q = -K\nabla p \cdot \mathbf{n}$, $\psi = \ln K$, $\nu = 1/K$, $\sigma = c\nu$,

Based on above equations, many researchers either process flux term q or add additional equations to the flux term. For 3D element, however, we find it infeasible anymore, therefore we reconsider the boundary integral equation after the element is applied.

$$-\lambda p_i + \int_\Gamma (p\nabla G \cdot \mathbf{n} - G\nabla p \cdot \mathbf{n}) ds + \iint_\Lambda G\left[-\nabla\psi \cdot \nabla p + \sigma\frac{\partial p}{\partial t} + \nu f\right] dA = 0 \tag{4}$$

We can find the treatment to flux term actually equals to the treatment to $\int_\Gamma G\nabla p \cdot \mathbf{n} ds$, therefore we can use the analog processing of finite volume method (FVM). Taking a 2D rectangular element as an example to demonstrate the process.

When we process the central element in Fig. 1, we assume node 1 is selected as the source, then

$$\int_\Gamma G\nabla p \cdot \mathbf{n} ds = \frac{p_6 - p_5}{\Delta y}\int_{\Gamma_1} G ds + \frac{p_7 - p_5}{\Delta x}\int_{\Gamma_2} G ds + \frac{p_8 - p_5}{\Delta x}\int_{\Gamma_1} G ds + \frac{p_9 - p_5}{\Delta y}\int_{\Gamma_1} G ds \tag{5}$$

Above treatment conforms to the primitive significance of the boundary integral equation, and makes the equations in one element contains all nodal values of the other elements which is neighboring to it. Besides, $\nabla p \cdot \mathbf{n}$ is included in the integral equation. It is evident that the integral operator is bounded, and when element size is small upper bound of the norm of the integral operator is also small, which helps effectively control estimation error of the method. Besides, as the decrease of element size, accuracy of the estimation of $\nabla p \cdot \mathbf{n}$ by difference quotient will be improved. As a result, the method theoretically enhances calculation accuracy. The method is evident to have a good quality of convergence, and error will be rapidly converged to zero as element size decreases, which is convenient for us to control error.