Engineering Analysis with Boundary Elements 95 (2018) 93–104

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Trefftz methods with cracklets and their relation to BEM and MFS

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ARTICLE INFO

MSC:
65N38
65N80

Keywords:
Trefftz method
Cracklets
Boundary element method
Method of fundamental solutions

ABSTRACT

In this paper we consider Trefftz methods which are based on functions defined by single layer or double layer potentials, integrals of the fundamental solution, or their normal derivative, on cracks. These functions are called *cracklets*, and satisfy the partial differential equation, as long as the crack support is not placed inside the domain. A boundary element method (BEM) interpretation is to consider these cracks as elements of the original boundary, in a direct BEM approach, or elements of an artificial boundary, in an indirect BEM approach. In this paper we consider the cracklets just as basis functions in Trefftz methods, as the method of fundamental solutions (MFS). We focus on the 2D Laplace equation, and establish some comparisons and connections between these methods with cracklets and standard approaches like the BEM, indirect BEM, and the MFS. Namely, we propose the enrichment of the MFS basis with the cracklets. Several numerical simulations are presented to test the performance of the methods, in particular comparing the results with the MFS and the BEM.

1. Introduction

The solution of boundary value problems (BVP) for partial differential equations (PDEs) benefits from the fact that for homogeneous linear PDEs, it is possible to write them as a linear combination of basis functions that satisfy the PDE, reducing the BVP to the fitting of the boundary conditions. This is a general context of the Trefftz methods in its different variants (e.g. [20]), and it has the advantage that no meshing procedure is needed for the domain.

The Boundary Element Method (BEM) has also a long history of application (e.g. [12]) and it may be included in the class of Trefftz methods, as it relies on boundary potentials that are solutions of the PDE inside the domain. The indirect BEM (IBEM) is another Trefftz variant, where the boundary potentials are considered on some artificial boundary surrounding the original one. The use of an artificial boundary is also the context of application of another Trefftz method - the method of fundamental solutions (MFS), no longer with boundary potentials, but simply as the location of point sources, i.e. fundamental solutions centered on source points, that are located on some artificial boundary (e.g. [2,13,14]).

A connection between some of these approaches was already pointed out in [10], and in this paper we explore this known connection further, using the concept of *cracklets* (e.g. [1]). Since the single and double layer potentials are defined on a boundary, which is decomposed in multiple boundary elements, we consider the elements of the boundary themselves to be cracks (or screens), and the cracklets are therefore defined as single or double layers over the support of a crack. If the boundary elements are seen as the union of cracks, then the BEM formulation may be understood with the use of cracklets on the boundary, and the IBEM formulation with the use of cracklets on some artificial domain.

However since these cracklets are themselves solutions of the PDE, with an appropriate behavior at infinity, they might be understood as a Trefftz basis, and we propose that they may be used with no direct connection to a boundary (artificial or original). In Section 3.5 we prove a density theorem and its corollary that justifies the completeness of this set of basis functions.

As in the MFS, a good location for the support of the cracklets, is an issue of current research (cf. [2,8,22]), and we either considered a standard approach using boundary dilation, like in the IBEM, or used a MFS choice as proposed in [2], or even considered cracklets on the original boundary as in the BEM, as was proposed in [5] to tackle difficulties in the approximation of discontinuous functions. In particular, to avoid the Gibbs oscillations when approximating discontinuous boundary conditions (e.g. [16]).

Several different methods have been considered to avoid the singularities associated with the fundamental solutions and allowing direct collocation on the boundary, such as the boundary knot method [15], the regularized meshless method [23], the modified method of fundamental solutions [21], or the singular boundary method [11]. In Section 3.7 we emphasize that when the solution does not have an

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https://doi.org/10.1016/j.enganabound.2018.06.021
Received 30 May 2018; Accepted 23 June 2018

analytic extension that goes beyond the analytic support of the approximation, the results will become worse. In fact it is known [7,18] that we can get exponential convergence of the MFS, as long as the solutions are entire, but the rate of convergence decreases fast, if the solutions do not extend analitically beyond the analytic support of the approximation, which is determined by the artificial boundary. In this sense the Trefftz method with cracklets proposed here is not attached to a specific artificial boundary as in an indirect boundary integral equation method. It inherits the ill conditioning of the inversion of compact operators (e.g. [19]), and these methods have been used to solve inverse problems, sharing the need of some regularization techniques (e.g. [4,17]).

The cracklets are here considered not only as a Trefftz method, but also as an enrichment technique for the MFS, for instance as proposed in [3,5,6], or to serve as a link between the MFS and the BEM. Recently, the method of angular basis functions (MABF) was proposed in [24,25] which considers double layer solutions, that are linked to angular measurements in the case of the Laplace equation. These functions are also considered in the fast implementation of BEM (e.g. [9]) and correspond to double layer cracklets.

In Section 2 we briefly recall the notions of layer potentials, and in Section 3 we consider the cracklet Trefftz method, as proposed for single and double layer potentials, but reduced to constant densities, and establish the main mathematical result that proves the completeness of the method. Finally in Section 4 we present numerical simulations that illustrate the performance of the different approaches.

2. Single and double layer potentials

This work will focus on harmonic boundary value problems, but it may be extended to problems such as

$$(P)\begin{cases} Du = 0, & \text{in } \Omega \\ Bu = g, & \text{on } \Gamma = \partial\Omega \end{cases} \tag{1}$$

where a fundamental solution Φ of the differential operator D is available.

In this paper, for simplicity, we assume $D = \Delta$ to be the 2D Laplace operator, with fundamental solution given by

$$\Phi(x) = \frac{1}{2\pi}\log||x||, \tag{2}$$

and when B is the identity operator, (P) is a Dirichlet boundary value problem, with unique solution $u \in H^1(\Omega)$, for a given $g \in H^{1/2}(\Gamma)$.

The corresponding single layer or double layer potentials (e.g. [19]) are respectively represented by $\mathcal{L}_\gamma$ and $\mathcal{M}_\gamma$,

$$\mathcal{L}_\gamma\alpha(x) = \int_\gamma \Phi(x-y)\alpha(y)ds_y \tag{3}$$

$$\mathcal{M}_\gamma\beta(x) = \int_\gamma \partial_{\boldsymbol{n}_y}\Phi(x-y)\beta(y)ds_y \tag{4}$$

where γ is a boundary that may coincide with Γ, $\alpha \in H^{-1/2}(\gamma)$ and $\beta \in H^{1/2}(\gamma)$ are unknown densities. Moreover, $\partial_{\boldsymbol{n}} = \nabla\cdot\boldsymbol{n}$ represents the normal derivative (we write $\partial_{\boldsymbol{n}_y}$ to be clear that the gradient is with respect to the y variable).

Remark 2.1. These layer potentials are defined for $x \notin \gamma$, and trace formulas may be obtained when $x \in \Omega \to x_\gamma \in \gamma$ (along the normal direction),

$$\mathcal{L}_\gamma\alpha(x) \to S_f\alpha(x_\gamma), \quad \mathcal{M}_f\beta(x) \to (\tau\mathcal{I} + \mathcal{K}_\gamma)\beta(x_\gamma) \tag{5}$$

where the expressions for S_f and $\mathcal{K}_\gamma$ are the same as for $\mathcal{L}_\gamma$ and $\mathcal{M}_\gamma$, respectively, but now with $x_\gamma \in \Gamma$, thus implying the definition of a (weakly) singular integral operator.

The parameter $\tau(x_\gamma) = \frac{1}{2}$ for all regular boundary points, but if x_γ is a corner point, then $\tau(x_\gamma) = \frac{\theta}{2\pi}$, where θ represents the internal angle (from 0 to 2π). Taking the trace from the exterior domain, with $x \notin \bar{\Omega}$, then τ has negative sign, corresponding to the external angle.

2.1. Integral equations

To find the unknown densities we consider the first kind integral equations when $x \in \Gamma$, and $\Gamma \neq \gamma$.

(i) Using the single layer potential, $\mathcal{L}_\gamma\alpha(x) = g(x)$, for $x \in \Gamma$, i.e.

$$\int_\gamma \Phi(x-y)\alpha(y)ds_y = g(x). \tag{6}$$

(ii) Using the double layer potential, $\mathcal{M}_\gamma\beta(x) = g(x)$, for $x \in \Gamma$, i.e.

$$\int_\gamma \partial_{\boldsymbol{n}_y}\Phi(x-y)\beta(y)ds_y = g(x). \tag{7}$$

Remark 2.2. When $\Gamma = \gamma$, we obtain

$$S\alpha(x) = g(x), \tag{8}$$

which is also a first kind integral equation on Γ, but a second kind integral equation is obtained for the double layer potential, since $(\tau\mathcal{I} + \mathcal{K})\beta(x) = g(x)$, which means

$$\tau(x)\beta(x) + \int_\Gamma \partial_{\boldsymbol{n}_y}\Phi(x-y)\beta(y)ds_y = g(x). \tag{9}$$

Remark 2.3. In the case of Neumann boundary conditions, the inner normal trace of the single and double layer potentials gives, when $x \in \Omega \to x_\gamma \in \gamma$ (along the normal direction),

$$\boldsymbol{n}(x_\gamma)\cdot\nabla\mathcal{L}_\gamma\alpha(x) \to (-\tau\mathcal{I} + \mathcal{K}'_\gamma)\alpha(x_\gamma) \tag{10}$$

$$\boldsymbol{n}(x_\gamma)\cdot\nabla\mathcal{M}_\gamma\beta(x) \to \mathcal{T}_\gamma\beta(x_\gamma) \tag{11}$$

with

$$\mathcal{K}'_\gamma\alpha(x_\gamma) = \int_\gamma \partial_{\boldsymbol{n}_x}\Phi(x_\gamma - y)\alpha(y)ds_y \quad \text{and}$$

$$\mathcal{T}_\gamma\beta(x_\gamma) = \int_\gamma \partial_{\boldsymbol{n}_x}\partial_{\boldsymbol{n}_y}\Phi(x_\gamma - y)\beta(y)ds_y.$$

The operator $\mathcal{T}_\gamma$ presents then a singular integration which is understood in the sense of the Cauchy principal value.

2.2. Direct and indirect boundary element approach

Consider the boundary element method in its two variants – the direct the and indirect approach.

(i) In the BEM, Eq. (8) or (9) are considered, taking $\gamma = \Gamma$.

(ii) In the IBEM, Eq. (6) or (7) are considered, with γ being an artificial boundary surrounding the original Γ. That is, we may take $\omega \supset \bar{\Omega}$ and $\gamma = \partial\omega$.

The integral equation (9) is of the second kind, it is better conditioned and usually preferred to the first kind integral equation (8).

The discretization of the integral operators on γ, equal or not to Γ, usually consists in splitting that boundary into boundary elements

$$\gamma = \gamma_1 \cup \cdots \cup \gamma_N \tag{12}$$

which may reproduce or approximate the full boundary. Thus,

$$\mathcal{M}\beta(x) = \sum_{n=1}^{N}\int_{\gamma_n} \partial_{\boldsymbol{n}_y}\Phi(x-y)\beta_n(y)ds_y \tag{13}$$

using local densities $\beta_n = \beta|_{\Gamma_n}$ and, for example, a trigonometric or a power series expansion, $\beta_n(s) = \beta_{n,0} + \beta_{n,1}s + \beta_{n,2}s^2 + \cdots$

In the simplest situation $\beta_n \cong \beta_{n,0}$ are constant terms, and the calculation resumes to the evaluation of

$$K_n(x) = \int_{\gamma_n} \partial_{\boldsymbol{n}_y}\Phi(x-y)ds_y \tag{14}$$

leading to the approximation

$$\mathcal{M}\beta(x) \approx \sum_{n=1}^{N}\beta_n K_n(x). \tag{15}$$