Engineering Analysis with Boundary Elements 95 (2018) 154–159

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

The adaptive algorithm for the selection of sources of the method of fundamental solutions

Ji Lin[a,*], A.R. Lamichhane[b], C.S. Chen[a,c], Jun Lu[d,e,*]

[a] International Center for Simulation Software in Engineering and Sciences, College of Mechanics and Materials, Hohai University, Nanjing 211100, China
[b] Department of Mathematics and Statistics, Ohio Northern University, Ada, OH 45810, USA
[c] Department of Mathematics, University of Southern Mississippi, Hattiesburg, MS 39406, USA
[d] Materials and Structural Engineering Departmentment, Nanjing Hydraulic Research Institute, Hujuguan Road No. 34, Nanjing 210024, China
[e] State Key Lab of Hydrology-Water Resources and Hydraulic Engineering, Xikang Road No. 1, Nanjing 210098, China

ARTICLE INFO

Keywords:
Method of fundamental solutions
Source nodes
Adaptive greedy algorithm
Meshless method

ABSTRACT

Despite all the efforts and success for finding the optimal location of the sources outside the domain for the method of fundamental solutions (MFS), this issue continues to attract the attention from researchers for seeking more efficient and reliable algorithms. In this paper, we propose to extend the adaptive greedy technique which applies the primal-dual formulation for the selection of source nodes in the MFS for Laplace equation with nonharmonic boundary conditions. Such approach is a data-dependent algorithm which adaptively selects the suitable source nodes based on the specific adaptive procedure. Both 2D and 3D examples are provided. Moreover, the proposed algorithm is easy to implement with high accuracy.

1. Introduction

The method of fundamental solutions (MFS) [1,2] is a meshless technique for the numerical solution of certain elliptic boundary value problems which falls in the class of methods generally called boundary-type methods [3,4]. The MFS has been considered as one of the most intuitive approaches to solve partial differential equations in science and engineering. In the MFS, the solution to the problem is represented by the linear combinations of fundamental solutions. To avoid the singularities of the fundamental solutions, the MFS places the source nodes away from the physical boundary. The main advantage of the MFS over domain discretization methods such as the finite element method and the finite difference method is that the approximation exactly satisfies the governing equation and one only have to enforce the approximation to satisfy the boundary conditions. The MFS also has some advantages over the boundary-type methods such as the boundary element method. The boundary discretization and integrations over the boundary are totally avoided in the MFS [5,6]. Until now, the MFS has been used to solve the steady-state heat conduction problem [7], the elasticity problem [8], the wave problem [9], and the bi-harmonic problem [10]. Some comprehensive reviews of the MFS and related methods are given in [11–13].

Although the MFS has been successfully used to solve engineering problems, one of the most arguable issue concerning the MFS is still the location of the source nodes. In the early stage of the MFS, the source nodes were taken as unknowns which yielded a nonlinear system of equations. One of the main disadvantages of this approach is computationally expensive. For such reasons, in the later applications, the locations of the source nodes are fixed in advance thus avoiding the nonlinearity and high computational cost. Usually, the number of collocation nodes is the same as the number of source nodes. The approximation of the MFS is given by the linear combination of fundamental solutions where the boundary conditions are satisfied by the collocation method, for which a system with a number of linear equations with an equal number unknowns is solved. Some improved version of the MFS in the literatures should be mentioned here. In [14] Smyrlis proposed the under-determined version of the MFS which selected more sources than collocation points. Karageorghis presented the Almansi of fundamental solutions for solving biharmonic problems where the boundary conditions were satisfied by the least square method [15]. A novel formulation of the MFS based on the simple layer potential representation of Fichera was proposed in [16]. Furthermore, in order to maintain the accuracy of the MFS approximation, the locations of the source nodes should be carefully determined. Smyrlis and Karageorghis indicated the poor accuracy of the method when the source nodes were placed either very close to or very far from the boundary [17]. Nishimura et al. presented the automatic arrangement of fictitious charges and contour points in charge simulation method for polar coordinate system [18]. A simulated annealing algorithm for optimising of the placement of the source nodes was studied in [19]. Adaptive refinement scheme for the least-squares approach of the method of fundamental solution for three-dimensional harmonic problems was proposed in [20] where a hierarchical data structure was used for spreading of the source nodes starting

* Corresponding author.
E-mail addresses: linji861103@126.com (J. Lin), lujunhhu1981@126.com (J. Lu).

https://doi.org/10.1016/j.enganabound.2018.07.008
Received 28 May 2018; Accepted 13 July 2018

from an initial coarse distribution. Within this approach, the number of source nodes was not fixed but some new sources are added by using a correction indicator that was related to, a sensitivity analysis of the solutions. Based on the numerical experiments with randomly distributed sources and theoretical analysis, Mitic indicated that the placements and number of source nodes were not quite important, providing that certain conditions were satisfied [21]. Nishimura and Nishimori adapted the immune algorithm to the 3D arrangement problem of fictitious charges and contour points [22]. Kołodziej and Klekiel proposed the evolutionary algorithms for optimization of method parameters in solution of Poisson's equation [23]. Gorzelańczyk and Kołodziej recommended geometrically similar contours instead of using domain decomposition in the case of torsion of prismatic rods on concave regions [24]. Alves [25] proposed a choice along the discrete normal direction with a local criterion to define the distance to the boundary. Gorzelańczyk et al., [26–28] applied the genetic algorithms for determining the arrangement of fictitious nodes in the MFS. The practical algorithm was proposed by Karageorghis [29] for determining the optimal pseudo-boundary. Chen et al. [30] suggested the leave-one-out cross validation algorithm for efficiently identifying the source nodes. Despite the efforts mentioned above, more attempts can be found in [31–34] and references therein. The algorithm proposed in [35] is the so-called greedy technique which is used to pick the radial basis function centers and collocation nodes [36]. Schaback [37] used the circular boundary and similar boundary points as the trial space for different bounded curves. This adaptive method tries to select the good source nodes by picking the column of the $m\times n$ matrix A whose multiples approximate the right hand side $m\times 1$ vector b in a best way and then transforms the problem to the space orthogonal to that column and repeats the process. Once the algorithm has selected a number of columns in this way, we take this column selection for a trial space and use the MFS for solving the given problem on that trial space. In this paper, we have used the adaptive greedy technique based on the primal-dual formulation of the minimum-norm solution to an over-determined system [38] to obtain the optimal source nodes. Also, the trial space is chosen in such a way that its elements lie on the exterior of the given curve and inside a bounded domain.

The rest of the paper is divided into four sections. In Section 2, we briefly introduce the general problem and the MFS formulation for its solution. In Section 3, we review the adaptive greedy technique used in this paper. Some numerical results are used to show the accuracy of the present method in Section 4. Section 5 draws some concluding remarks.

2. The method of fundamental solutions

Since the present approach can be extended to general equations, we consider the following simple Laplace equation under the Dirichlet boundary condition as an example

$$\Delta u = 0,\ \texttt{in}\ \Omega, \tag{1}$$

where Δ is the Laplace operator, u is the unknown function to be determined, and Ω represents the bounded simply connected domains in 2D and 3D. Eq. (1) subjects to the following Dirichlet boundary condition

$$u = g,\ \texttt{on}\ \partial\Omega, \tag{2}$$

where the function g is given in advance and doesn't necessarily to be a harmonic function.

In the MFS, the solution to Eqs. (1) and (2) is given by the linear combination of the fundamental solutions

$$u_N(\boldsymbol{x}) = \sum_{k=1}^{N} \lambda_k G(\boldsymbol{x}, \boldsymbol{\xi}_k), \boldsymbol{x} \in \Omega, \tag{3}$$

where $\{\boldsymbol{\xi}_k\}_{k=1}^N$ are the source nodes placed away from the boundary, $\{\lambda_k\}_{k=1}^N$ are unknown coefficients to be determined by the collocation method, and the $G(\boldsymbol{x}, \boldsymbol{\xi})$ is the fundamental solution of the Laplace equation, as follows

$$G(\boldsymbol{x}, \boldsymbol{\xi}) = -\frac{1}{2\pi} \ln \|\boldsymbol{x} - \boldsymbol{\xi}\|_2, \tag{4}$$

for two dimensional problems, and

$$G(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{1}{4\pi \|\boldsymbol{x} - \boldsymbol{\xi}\|_2}, \tag{5}$$

for three dimensional problems.

In the collocation method, we also take a set of collocation nodes $\{\boldsymbol{x}_l\}_{l=1}^M$ on the boundary $\partial\Omega$. Collocating the given boundary condition will yield

$$u_N(\boldsymbol{x}_l) = g(\boldsymbol{x}_l),\ l = 1,\ \ldots,\ M. \tag{6}$$

The resulting matrix expression of the MFS discretization Eq. (6) can be given in the following matrix form

$$\boldsymbol{A}\lambda = \boldsymbol{g}, \tag{7}$$

where $\boldsymbol{A}$ is a $M\times N$ matrix and $\boldsymbol{g}$ is a $M\times 1$ vector defined by

$$\boldsymbol{A}_{l,k} = G(\boldsymbol{x}_l, \boldsymbol{\xi}_k),\ \boldsymbol{g}_l = g(\boldsymbol{x}_l),\ l = 1, \ldots, M, and,\ k = 1, \ldots, N, \tag{8}$$

and the unknown λ can be determined if $M \geq N$.

3. The adaptive greedy technique

In order to use the adaptive greedy technique, we take the number of source nodes larger than the number of boundary collocation nodes which means that $M \leq N$. Consider the under-determined system of Eq. (7) with $\boldsymbol{A} \in R^{M\times N}$, $\boldsymbol{g} \in R^M$, and $M \leq N$. Then the solution to the under-determined system is defined through the following minimization problem:

$$\texttt{min}\ \frac{1}{2}\lambda^T I \lambda,\ \texttt{subject to},\ \boldsymbol{A}\lambda - \boldsymbol{g} = 0. \tag{9}$$

The primal-dual formulation of Eq. (7) can be given in the following sub-matrices form using the method of Lagrange multipliers, as follows:

$$\begin{bmatrix} I & \boldsymbol{A}^T \\ \boldsymbol{A} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \lambda \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{g} \end{bmatrix}, \tag{10}$$

where I is the identity matrix and $\boldsymbol{v}$ is the vector of Lagrange multipliers. The adaptive algorithm for the determination of the nearly optimal source nodes is depended on the system of Eq. (10). For the adaptive algorithm, different sub-matrices of the coefficient matrix $\boldsymbol{A}$ and $\boldsymbol{g}$ are required. We define the sub-matrices of the matrix $\boldsymbol{A}$ by $\boldsymbol{A}(\cdot)$: $R^m \times R^n \to R^{m\times n}$ as the sub-matrices made up of the m number of boundary collocation nodes and n number of source nodes. Similar, the sub-vectors of $\boldsymbol{g}$ by $\boldsymbol{g}(\cdot)$: $R^m \to R^m$ which are the boundary values of the chosen boundary collocation nodes.

In the adaptive algorithm, a new indexed sets is used which is denoted by $X_k = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k\}$ and $\zeta_k = \{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_k\}$, $k = 1, 2, \ldots, M$ for the boundary collocation nodes and source nodes, respectively, such that the chosen nodes are placed earlier in the given lists. Suppose that, after k iterations, the adaptive algorithm has chosen a set of k boundary collocation nodes $X_k \subset X_M$, and a set of source nodes $\zeta_k \subset \zeta_N$, respectively. These chosen boundary nodes and sources define a sub-problem to the original problem, as follows:

$$\begin{aligned} \boldsymbol{A}_k \hat{\lambda}^k &= \hat{\boldsymbol{g}}^k, \\ \boldsymbol{A}_k^T \hat{\boldsymbol{v}}^k &= -\hat{\lambda}^k, \end{aligned} \tag{11}$$

where $\boldsymbol{A}_k = \boldsymbol{A}(X_k, \zeta_k)$ and $\hat{\boldsymbol{g}}^k = \boldsymbol{g}(X_k)$ which are the coefficient matrix and boundary conditions defined by X_k boundary collocation nodes and ζ_k source nodes. After solving the Eq. (11), the $\lambda^k \in R^N$ can be the extension of the obtained $\hat{\lambda}^k$ by adding zeros into the sequence according to the non-selected source nodes. Similarly, $\boldsymbol{v}^k \in R^N$ can be the extension of the obtained $\hat{\boldsymbol{v}}^k$ by adding zeros. The primal PDE residual, with respect to the intermediate solution $\lambda_{\boldsymbol{k}}$ can be obtained by

$$\boldsymbol{r}^k = \boldsymbol{A}\lambda^k - \boldsymbol{g} = \boldsymbol{A}(X_M, \zeta_k)\hat{\lambda}^k - \boldsymbol{g}. \tag{12}$$