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Corner restrictions and their application to bending plate analyses by the boundary element method

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ABSTRACT

In this paper, a study of the boundary restrictions at corner plates is presented for analysis by the boundary element method. From a hypothesis regarding the uniqueness of the stress tensor and algebraic manipulation, the equivalent shear force and bending moment were shown to be null for some types of corners.

With the imposition of these boundary conditions, the results obtained using the boundary element method are highly consistent with those obtained by the finite element method with very refined meshes.

Ways to write additional equations for each corner of the plate when the traction and its respective displacement are zero are also presented to avoid singularity in the system of equations.

1. Introduction

The boundary element method (BEM) was developed after the so-called domain methods, the Finite Difference Method (FDM) and the Finite Elements Method (FEM), already had consolidated formulations and broad applications. Its primary features, which were shown from the first formulations, are reductions in the approximations involved in any numerical analysis, the reduction of the systems of linear equations to be solved and the reduction and simplification of the input data. All these characteristics are basically due to a reduction in the dimension of the problem. For example, in a three-dimensional problem, the analysis will depend on the study of its surface.

With respect to the application of the BEM to plate analysis, the development of the method is based on the works of Jaswon et al. [1], who proposed a solution via integral equations of bi-harmonic equations, and they later applied it to the plate solution. In this formulation, the bi-harmonic equation is decomposed into two harmonic equations that are solved by integral equations. Other works that can be cited are Hansen [2], Stern [3], Bezini [4,5], Oliveira Neto and Paiva [6,7], Paiva and Mendonça [8], Paiva and Aliabad [9] and Paiva and Venturini [10,11].

In applying the BEM to the plate bending analysis, the boundary is divided into segments called boundary elements, and approximation functions are adopted for the displacements and tractions in the domain of each element. The first option is to use the constant element, that is, the displacements and tractions at the boundary of the plate are assumed to be constant in the domain of each element, and a single node is located on its midpoint. As in the boundary integral equations for plate bending analysis, when the plate corner reactions appear, there are additional equations for each plate corner that are usually located at the end of the system of equations that is obtained for the variables associated with the element nodes. The second option is to use the linear boundary element, with linear approximation for displacements and tractions in the domain of each element. In this case, two nodes are associated, usually at the ends of each element and double nodes appear in the corners, with the same coordinates but at different sides of the corner.

In the BEM, the equivalent shear force $V_{n,}$ and the bending moment m_n appear as input data or unknowns of the system of equations. The traction in the corners can be written as a function of the displacements and the derivatives in its sides. Thus, regardless of the corner angle value, traction is achieved through the solution of the algebraic equation system. Thus, regardless of the analytical solution for the plate in question, the bonding conditions and the respective traction must be obeyed. Clearly, for corners in which the angle is greater than $\pi/2$, stress concentrations may occur in the plate corner neighbourhood, and they may directly influence the boundary tractions causing an oscillation in their values in their vicinity. The same influence does not occur with the finite element method. Because the equivalent shear force is calculated as concentrated forces at the nodes of the finite elements and the bending moment is calculated from the finite element form functions using the displacements obtained during the plate analysis, the influence of the stress concentration is more significant.

Because the equivalent shear force and the bending moment for the BEM in the normal direction on the side of the plate appear as unknowns when the respective displacements and their derivative are zero, the boundary conditions relating to these tractions in the plate corner are of fundamental importance in the results and in their distribution along the plate side.

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Supposing that there is a non-zero equivalent shear force in corners where it does not exist will result in a distribution of this traction along the side of the plate with large oscillations, and only at points far from the corner will the results approach the solution of the problem. The same result occurs when supposing the existence of non-zero corner reactions in corners where they do not exist. Additionally, the assumption of the existence of a bending moment in the normal direction to the side where it does not exist results in a bending moment distribution with large oscillations.

In spite of the problems associated with the corner discussed above, even if the bending moment and the shear force exhibit large oscillations, the results in the plate domain, displacements and bending moments are obtained with good precision.

We have 12 variables in a double node plate corner. For example, in a plate corner with the double nodes i and j, we have the following variables: bending moment and derivative in the normal direction (m_{ni} and θ_{ni}, m_{nj} and θ_{nj}), the equivalent shear forces and their respective displacements in the corner (V_{ni} and w_i; V_{nj} and w_j) and the components of the corner reaction (m_{nsi} and w_i; m_{nsj} and w_j). The latter factor could be concentrated into one factor directly involving the corner reaction and the respective displacement. However, this concentration cannot be performed if the adopted formulation admits double nodes in the corner of the plate. The imposition of boundary conditions referring to the bending moment m_n and its rotation do not present problems from the perspective of assembling the system of equations. However, in relation to the other variables, the same result does not occur. Let us initially admit a plate corner without a corner reaction and having the displacements in these nodes prescribed and equal to one another. After imposing these boundary conditions, the equations of the corner variables, now the equivalent shear force at nodes i and j, are equal to one another, leading to singularity in the system of equations. If we also consider the corner reaction and the same displacements at nodes i and j, we are including two new unknowns with the same number of equations. A solution that is usually adopted is to use the discontinuous element for the corner elements, bringing the corner nodes into the elemental domain and then an extra equation associated with an extra variable, and the corner node equations can now be written. However, this solution did not provide accurate results.

One of the problems involved in this analysis is the corner reaction for which its consideration may or may not lead to instabilities in the equivalent shear force, when it exists. The primary reason why the above formulations are not giving good results is that not every plate corner has a reaction. Marcus [12] has shown that the corner reaction only exists for simple supported corners whose angle is $\pi/2$. For example, in the case of a triangular plate that was simply supported on the boundary, with uniformly distributed loading [13]. In this study, we obtain the same result as Marcus [12], but with a different approach, and it is extended to plate corners with other boundary conditions. In this study, it was assumed that the sides of the plate corner are not subject to a distributed moment, which would lead to different results from those found here.

Some solutions have been presented for this problem. For example, Katsikadelis [14] utilized a constant boundary element and did not write the corner boundary equations for plate corners whose sides are supported vertically. From the results obtained here and using finite differences, the plate corner reaction is obtained. However, it is well known that finite differences always involve some imprecision and can lead to errors in the calculation of the corner reaction, for example, reaching a value other than zero when this does not occur. In addition, this formulation requires a large number of boundary elements to give good results. In some types of plate corners, the equivalent shear force and the bending moment are also equal to zero. If these conditions are not imposed in the system of equations obtained by the BEM, the equivalent shear force and the bending moment at the corner of the plate and its distributions along the side of the plate present huge instabilities and can approach the solution of the problem only in points far from the corner. In Paiva [15], the equivalent shear force V_n is approximated by applying concentrated reactions to the element nodes. Because the corner reactions act on the nodes that have already been considered, their values are also represented by these concentrated reactions.

It is important to note that both the constant and discontinuous linear boundary element and others of higher degrees give good results for displacements and moments at points in the plate domain that are not very close to its boundary. However, the number of boundary elements must be greater for the constant element than for the element with cubic approximation. The greatest error in the results obtained using the various formulations is in the boundary tractions, namely, the equivalent shear force and the bending moment. A large oscillation occurs in the distribution of these tractions along the sides of the plate for the constant element and for the linear ones, without considering the double node. For these boundary tractions, the type of boundary element has a huge influence on the results.

In this paper, based on the hypothesis on the uniqueness of the stress tensor and by using algebraic manipulations, a study is presented on the plate corner, showing under which conditions the equivalent shear force and bending moment are null. The results by Marcus [12] for the plate corner reaction are also confirmed with this approach. Ways to write additional equations for each corner of the plate are also presented for when the traction and its respective displacement are zero, avoiding singularity in the system of equations.

Plates with different combinations of corner restrictions are analysed, and the results are compared with those obtained with the Finite Element Method. The results show excellent consistency.

2. Integral equations

For a plate during bending, the following boundary integral equations can be written, using the alternative formulation of the boundary element method with three nodal displacement parameters [6]:

$$K(S)w(S)+\int_{\Gamma}\left[q_n^*(S,Q)w(Q)-m_n^*(S,Q)\frac{\partial w}{\partial n}(Q)-m_{ns}^*(S,Q)\frac{\partial w}{\partial s}(Q)\right]d\Gamma(Q)$$
$$=\int_{\Gamma}\left[V_n(Q)w^*(S,Q)-m_n(Q)\frac{\partial w^*}{\partial n}(S,Q)\right]d\Gamma(Q)+\sum_{i=1}^{N_c}R_{ci}(Q)w_{ci}^*(S,Q)$$
$$+\int_{\Omega_g}g(q)w^*(S,q)d\Omega_g(q) \tag{1}$$

where $\boldsymbol{w}$, $\boldsymbol{m_n}$ and $\boldsymbol{V_n}$ are the transverse displacement, the bending moment and the equivalent shear force, respectively, along the boundary; and $\boldsymbol{g(q)}$ and $\boldsymbol{\Omega_g}$ are the transverse load and the surface where it is applied. The symbol * is used here to indicate the fundamental solution. In this equation,

$K(s)=1$ for internal points s;

$K(S)=\beta/2\pi$ for a point S at a boundary corner, with internal angle β;

$K(S)=1/2$ for a point S on a smooth boundary;

$R_{ci}=m_{ns}^- - m_{ns}^+$ is the corner reaction;

From Eq. (1), the integral representation of the derivative for the displacement with respect to direction $\boldsymbol{m_s}$ of a system of coordinates $\boldsymbol{(m_s, u_s)}$ *located at any point of the plate* can be derived as follows:

$$K_1(S)\frac{\partial w}{\partial m_S}(S)+K_2(S)\frac{\partial w}{\partial u_S}(S)+\int_{\Gamma}\left[\frac{\partial q_n^*}{\partial m_s}(S,Q)w(Q)\right.$$
$$\left.-\frac{\partial m_n^*}{\partial m_S}(S,Q)\frac{\partial w}{\partial n}(Q)-\frac{\partial m_{ns}^*}{\partial m_S}(S,Q)\frac{\partial w}{\partial s}(Q)\right]d\Gamma(Q)$$
$$=\int_{\Gamma}\left\{V_n(Q)\frac{\partial w^*}{\partial m_S}(S,Q)-m_n(Q)\frac{\partial}{\partial m_S}\left[\frac{\partial w^*}{\partial n}(S,Q)\right]\right\}d\Gamma(Q)$$
$$+\sum_{i=1}^{N_c}R_{ci}(Q)\frac{\partial w_{ci}^*(S,Q)}{\partial m_s}+\int_{\Omega_g}g(q)\frac{\partial w^*}{\partial m_s}(S,q)d\Omega_g(q) \tag{2}$$

in which