Engineering Analysis with Boundary Elements 95 (2018) 25–32

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Schemes in generalized finite differences for seismic wave propagation in Kelvin–Voight viscoelastic media

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ARTICLE INFO

Keywords:
Meshless methods
Generalized finite differences method
Seismic waves
Viscoelasticity

ABSTRACT

Seismic wave propagation in homogeneous and isotropic Kelvin–Voight viscoelastic media is dealt with the meshless generalized finite difference method. The schemes in generalized finite differences for the decoupled system P-SV and SH are obtained. For each scheme, a stability limit is achieved and the star dispersion is calculated. Some cases are shown using irregular discretizations.

1. Introduction

Viscoelastic media have been considered for the simulation of seismic wave propagation by Day and Minster [1] who approximated a viscoelastic modulus by a rational function, coefficients of which are obtained by the Pad approximate method. Emmerich and Korn [2] used Generalized Maxwell Body (GMB-EK) as a relaxation function. Moreover, Carcione et al. [3], in accordance with Liu et al. [4], developed a method that used the Generalized Zener Body (GZB). Many authors such as Moczo have used the GMB-EK and others, such as Robertsson, chose to use the GZB theory, and then, two parallel lines of development were established until Moczo and Kristek [5] showed the relationship between two rheologies.

Attenuation is described adequately by means of the Kelvin–Voight viscoelastic model in many media at ambient temperature, as seen in Auld [6]. It is clear that the Kelvin–Voight model is a particular case, but still it is used in most applications [7]. Some of these applications are related to seismic exploration or earthquake seismology [8]. As an example, Sahu et al. [9] studied the gravity effect, the internal friction and the heterogeneity of media by means of SH wave propagation using the Kelvin–Voight constitutive relation.

Most authors, such as Carcione [10] or Moczo et al. [11], deal with memory variables to avoid the implementation of the constitutive law given by the convolution relation. The Kelvin–Voight model can be obtained as a particular case of relaxation function in the formulation with memory variables [3]. In spite of this, the Kelvin–Voight model has the advantage of not requiring additional variables with the corresponding decrease in the computational cost as Carcione et al. [8] point out.

The scheme in finite differences for SH waves in viscoelastic media with the Kelvin–Voight constitutive relation is shown in Kalyani and Chakraborty [12]. The same authors emphasize the superiority of the finite differences method to model seismic waves propagation as a consequence of its power, rapidity, flexibility and accuracy. A drawback of the finite differences method is the dependence of a regular grid that can complicate the discretization in complex geometries. This disadvantage can be overcome using the generalized finite differences method.

The generalized finite differences method (GFDM) is a meshless numerical method that provides values of the partial derivatives at each point (node) of a domain and allows us to solve problems easily in any geometry.

This method has already been applied to other kind of problems, as for example, in wave propagation in elastic media to deal with interfaces [13], to obtain the two-dimensional shallow water equations [14], to simulate the two-dimensional sloshing phenomenon[15], in thermoelastic analysis [16–19] or in inverse heat source problems [20].

This paper is focused on obtaining the schemes in generalized finite differences for the P-SV and SH wave equations in a Kelvin–Voight viscoelastic medium, obtaining a limit for the stability, obtaining a formula for measuring the star dispersion and obtaining the equations for Neumann boundaries.

The obtained schemes are applied in order to compare the numerical and analytical results in some academic cases. The application of the method is highlighted in adverse conditions of irregularity in each case. Nevertheless, in no event is it intended to analyze the influence of parameters or to solve specific practical problems. These issues will be the object of future works.

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https://doi.org/10.1016/j.enganabound.2018.06.017
Received 15 March 2018; Received in revised form 15 June 2018; Accepted 17 June 2018

Therefore, the paper is organized as follows. The schemes in GFD are presented in Section 2, the analysis of stability in Section 3 and the analysis of star dispersion in Section 4. In Section 5 the method is applied in irregular discretizations.

2. Kelvin–Voight viscoelastic waves in GFD

2.1. Wave equations in viscoelastic media

Let us consider the x–z plane and the decoupled problem for P-SV and SH waves. Let U, V and W be the displacements. As in Ben-Menahem and Singh [21], the correspondences $\lambda \to \lambda + \lambda' \frac{\partial}{\partial t}$ and $\mu \to \mu + \mu' \frac{\partial}{\partial t}$ are used in order to establish the constitutive equations of a linear viscoelastic media following a Kelvin model.

$$\text{P-SV}\begin{cases} \sigma_{xx} = \left(\lambda + \lambda' \frac{\partial}{\partial t}\right)(U_{,x} + W_{,z}) + 2\left(\mu + \mu' \frac{\partial}{\partial t}\right)U_{,x} \\ \sigma_{zz} = \left(\lambda + \lambda' \frac{\partial}{\partial t}\right)(U_{,x} + W_{,z}) + 2\left(\mu + \mu' \frac{\partial}{\partial t}\right)W_{,z} \\ \sigma_{xz} = \left(\mu + \mu' \frac{\partial}{\partial t}\right)(U_{,z} + W_{,x}) \end{cases} \quad (1)$$

$$\text{SH}\begin{cases} \sigma_{xy} = \left(\mu + \mu' \frac{\partial}{\partial t}\right)V_{,x} \\ \sigma_{yz} = \left(\mu + \mu' \frac{\partial}{\partial t}\right)V_{,z} \end{cases} \quad (2)$$

being λ and μ the Lamé parameters, ρ the density and λ' and μ' the viscosity coefficients.

The general equations of motion expressed in terms of displacements for both P-SV and SH waves are

$$\text{P-SV}\begin{cases} U_{,tt} = \alpha^2 U_{,xx} + \beta^2 U_{,zz} + (\alpha^2 - \beta^2)W_{,xzt} + \tau_P \alpha^2 U_{,xxt} \\ \qquad + \tau_S \beta^2 U_{,zzt} + (\tau_P \alpha^2 - \tau_S \beta^2)W_{,xzt} \\ W_{,tt} = \alpha^2 W_{,zz} + \beta^2 W_{,xx} + (\alpha^2 - \beta^2)U_{,xz} + \tau_P \alpha^2 W_{,zzt} \\ \qquad + \tau_S \beta^2 W_{,xxt} + (\tau_P \alpha^2 - \tau_S \beta^2)U_{,xzt} \end{cases} \quad (3)$$

$$\text{SH}\{V_{,tt} = \beta^2(V_{,xx} + V_{,zz}) + \tau_S \beta^2(V_{,xxt} + V_{,zzt}) \quad (4)$$

where α and β are the velocities of P and S waves, respectively, and $\tau_P = \frac{\lambda' + 2\mu'}{\lambda + 2\mu}$ and $\tau_S = \frac{\mu'}{\mu}$ are the relaxation times for P and S waves, respectively.

2.2. Formulae in generalized finite differences

Let us consider a discretization M of a domain D, the interior node $(\mathbf{x}_0, t_n) \in \Omega = M \cap int(D)$, the star $E(\mathbf{x}_0)$ with $N+1$ nodes and the function Ψ. This method allows to obtain the values of the partial derivatives in each node by means of a Taylor expansion and using a moving least squares approximation. The obtained function is

$$B(\mathbf{D}_\psi) = \sum_{i=1}^{N} (\psi_0 - \psi_i + \boldsymbol{\epsilon}_i^T \mathbf{D}_\psi)^2 w_i^2 \quad (5)$$

where w_i is a weighting function, ψ is an approximation of Ψ and

$$\mathbf{D}_\psi = \begin{pmatrix} \psi_{0,x} & \psi_{0,z} & \psi_{0,xx} & \psi_{0,xz} & \psi_{0,zz} \end{pmatrix}^T \quad (6)$$

$$\boldsymbol{\epsilon}_i = \begin{pmatrix} h_i & k_i & h_i^2/2 & h_i k_i & k_i^2/2 \end{pmatrix}^T \quad (7)$$

with h_i and k_i relative coordinates regarding the central node.

By minimizing (5) the values of the partial derivatives are obtained. So if m_{0pq} denotes the coefficient of the central node and m_{spq} denotes the coefficient of each node in the rest of the star, p, $q \in \{x, z\}$ and $s \in \{1, \dots N\}$, then the partial derivatives are given by

$$\psi_{0,pq} = -m_{0pq}\psi_0 + \sum_{s=1}^{N} m_{spq}\psi_s \quad (8)$$

More details can be seen in Benito et al. [22] or in M. Ureña et al. [23].

2.3. Schemes in generalized finite differences

The temporal approximations are given by the following classical finite differences for first and second order, respectively

$$\psi_{0,t}^n = \frac{3\psi_0^n - 4\psi_0^{n-1} + \psi_0^{n-2}}{2 \triangle t} \quad (9)$$

$$\psi_{0,tt}^n = \frac{\psi_0^{n-1} - 2\psi_0^n + \psi_0^{n+1}}{\triangle t^2} \quad (10)$$

These schemes can be found in [24]. They have been used for consistency with the rest of the schemes in the formulae as they have second order.

Substituting (8), (9) and (10) in (3) and (4), and arranging terms, the schemes for P-SV and SH waves in viscoelastic media are achieved

$$\text{P-SV}\begin{cases} u_0^{n+1} = (2 - A_{u_0}^0)u_0^n + (A_{u_0}^{-1} - 1)u_0^{n-1} - A_{u_0}^{-2}u_0^{n-2} \\ \qquad + \sum_{k=0}^2 \left[(-1)^{k+1} A_{w_0}^{-k} w_0^{n-k} + \sum_{j=1}^N (-1)^k (A_{u_j}^{-k} u_j^{n-k} + A_{w_j}^{-k} w_j^{n-k})\right] \\ w_0^{n+1} = (2 - B_{w_0}^0)w_0^n + (B_{w_0}^{-1} - 1)w_0^{n-1} - B_{w_0}^{-2}w_0^{n-2} \\ \qquad + \sum_{k=0}^2 \left[(-1)^{k+1} B_{u_0}^{-k} u_0^{n-k} + \sum_{j=1}^N (-1)^k (B_{w_j}^{-k} w_j^{n-k} + B_{u_j}^{-k} u_j^{n-k})\right] \end{cases} \quad (11)$$

$$\text{SH}\begin{cases} v_0^{n+1} = (2 - C_{v_0}^0)v_0^n + (C_{v_0}^{-1} - 1)v_0^{n-1} - C_{v_0}^{-2}v_0^{n-2} \\ \qquad + \sum_{k=0}^2 \sum_{j=1}^N (-1)^k C_{v_j}^{-k} v_j^{n-k} \end{cases} \quad (12)$$

where, for $j \in \{0, 1, 2, \dots, N\}$, the coefficients A, B and C are

$$\begin{cases} A_{u_j}^0 = \varphi_1^P \alpha^2 m_{jxx} + \varphi_1^S \beta^2 m_{jzz} \\ A_{u_j}^{-1} = \varphi_2^P \alpha^2 m_{jxx} + \varphi_2^S \beta^2 m_{jzz} \\ A_{u_j}^{-2} = \varphi_3^P \alpha^2 m_{jxx} + \varphi_3^S \beta^2 m_{jzz} \\ A_{w_j}^0 = B_{u_j}^0 = (\varphi_1^P \alpha^2 - \varphi_1^S \beta^2) m_{jxz} \\ A_{w_j}^{-1} = B_{u_j}^0 = (\varphi_2^P \alpha^2 - \varphi_2^S \beta^2) m_{jxz} \\ A_{w_j}^{-2} = B_{u_j}^0 = (\varphi_3^P \alpha^2 - \varphi_3^S \beta^2) m_{jxz} \\ B_{w_j}^0 = \varphi_1^P \alpha^2 m_{jzz} + \varphi_1^S \beta^2 m_{jxx} \\ B_{w_j}^{-1} = \varphi_2^P \alpha^2 m_{jzz} + \varphi_2^S \beta^2 m_{jxx} \\ B_{w_j}^{-2} = \varphi_3^P \alpha^2 m_{jzz} + \varphi_3^S \beta^2 m_{jxx} \\ C_{v_j}^0 = \varphi_1^S \beta^2 (m_{jxx} + m_{jzz}) \\ C_{v_j}^0 = \varphi_2^S \beta^2 (m_{jxx} + m_{jzz}) \\ C_{v_j}^0 = \varphi_3^S \beta^2 (m_{jxx} + m_{jzz}) \end{cases} \quad (13)$$

and, for $\star \in \{P, S\}$,

$$\begin{aligned} \varphi_1^\star &= \triangle t^2 + 1.5\tau_\star \triangle t \\ \varphi_2^\star &= 2\tau_\star \triangle t \\ \varphi_3^\star &= 0.5\tau_\star \triangle t \end{aligned} \quad (14)$$

2.4. Boundary conditions

Dealing with Dirichlet conditions is straightforward. Dealing with Neumann conditions requires a more careful approach. In this last case, it is necessary to add new nodes outside the domain and to solve the system of equations given by means of the suitable discretization of the equations in the free surface as in Benito et al. [25] for P-SV waves or